

# CHERN CLASSES AND COMPATIBLE POWER OPERATIONS IN INERTIAL K-THEORY

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ABSTRACT. We develop a theory of Chern classes and compatible power operations for *strongly Gorenstein inertial pairs* (in the sense of [EJK12]).

An important application is to show that there is a theory of Chern classes and compatible power operations for the virtual product defined by [GLS<sup>+</sup>07]. We also show that when  $\mathcal{X}$  is a quotient  $\mathcal{X} = [X/G]$ , with  $G$  diagonalizable, then inertial K-theory of  $\mathcal{X}$  has a  $\lambda$ -ring structure. This implies that for toric Deligne-Mumford stacks there is a corresponding  $\lambda$ -ring structure associated to virtual K-theory.

As an example we compute the semi-group of  $\lambda$ -positive elements in the virtual  $\lambda$ -ring of the weighted projective stacks  $\mathbb{P}(1, 2)$  and  $\mathbb{P}(1, 3)$ . Using the virtual orbifold line elements in this semi-group, we obtain a simple presentation of the K-theory ring with the virtual product and a simple description of the virtual first Chern classes. This allows us to prove that the completion of this ring with respect to the augmentation ideal is isomorphic to the usual K-theory of the resolution of singularities of the cotangent bundle  $\mathbb{T}^*\mathbb{P}(1, 2)$  and  $\mathbb{T}^*\mathbb{P}(1, 3)$ , respectively. We interpret this as a manifestation of mirror symmetry, in the spirit of the Hyper-Kähler Resolution Conjecture.

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## 1. INTRODUCTION

Beginning with the pioneering work of Chen and Ruan [CR02], there is a well developed theory of orbifold products associated to the inertia stack  $I\mathcal{X}$  of a smooth Deligne-Mumford stack  $\mathcal{X}$ . In particular, there are orbifold products for the cohomology, Chow groups and K-theory of  $I\mathcal{X}$ . Moreover, there is an orbifold Chern

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character  $\mathcal{Ch}: K(I\mathcal{X}) \rightarrow A^*(I\mathcal{X})_{\mathbb{Q}}$  which respects these products [JKK07]. In this paper we are motivated by mirror symmetry to consider the further question of whether there is a corresponding theory of *orbifold Chern classes*. Since the classical theory of *power operations* in K-theory is tightly bound to the theory of Chern classes, we investigate the question of whether there are also corresponding operations on orbifold K-theory.

We prove the following result about quotient orbifolds  $\mathcal{X} = [X/G]$ .

### Main Results.

- (a) *If the orbifold  $\mathcal{X}$  is Gorenstein, then there is an orbifold Chern class homomorphism  $c_t: K(I\mathcal{X}) \rightarrow A^*(I\mathcal{X}) \otimes \mathbb{Q}[[t]]$  (see Definition 5.1 and Theorem 5.13).*
- (b) *If the orbifold  $\mathcal{X}$  is strongly Gorenstein (see below for definition), then there are Adams  $\psi$  and  $\lambda$  operations compatible with the Chern class homomorphism (see Definitions 5.3 and 5.5 as well as Theorem 5.13).*
- (c) *If  $G$  is diagonalizable and  $\mathcal{X}$  is strongly Gorenstein, then the Adams and  $\lambda$  operations make  $K(I\mathcal{X}) \otimes \mathbb{Q}$  with its orbifold product into a rationally augmented  $\lambda$ -ring (see Theorem 5.16).*
- (d) *If the orbifold  $\mathcal{X}$  is strongly Gorenstein, then there is an inertial dual operation  $\mathcal{F} \rightarrow \mathcal{F}^\dagger$  on  $K(\mathcal{X})$  which is an involution and a ring homomorphism and which commutes with the orbifold Adams operations and the orbifold augmentation. (See Theorem 6.3)*

Our method of proof is based on developing properties of *inertial pairs* defined in [EJK12]. An inertial pair  $(\mathcal{R}, \mathcal{S})$  consists of a vector bundle  $\mathcal{R}$  on the double inertia stack  $\mathbb{I}^2\mathcal{X}$  together with a class  $\mathcal{S} \in K(\mathcal{X})_{\mathbb{Q}}$  satisfying certain compatibility conditions. The bundle  $\mathcal{R}$  determines associative *inertial products* on  $K(I\mathcal{X})$  and  $A^*(I\mathcal{X})$ , and the class  $\mathcal{S}$  determines both a rational grading on  $A^*(I\mathcal{X})$  and a Chern character homomorphism of inertial rings  $\mathcal{Ch}: K(I\mathcal{X}) \rightarrow A^*(I\mathcal{X})_{\mathbb{Q}}$ .

The basic example of an inertial pair  $(\mathcal{R}, \mathcal{S})$  is the orbifold obstruction bundle  $\mathcal{R}$  together with the class  $\mathcal{S}$  defined in [JKK07] and considered in [EJK10], but this is far from being the only example. Each vector bundle  $V$  on  $\mathcal{X}$  determines two inertial pairs,  $(\mathcal{R}^+V, \mathcal{S}^+V)$  and  $(\mathcal{R}^-V, \mathcal{S}^-V)$ . If we denote the tangent bundle of  $\mathcal{X}$  by  $\mathbb{T}$ , then the inertial pair  $(\mathcal{R}^-\mathbb{T}, \mathcal{S}^-\mathbb{T})$  produces the virtual orbifold product of [GLS<sup>+</sup>07].

An inertial pair  $(\mathcal{R}, \mathcal{S})$  is called *Gorenstein* (*strongly Gorenstein*) if  $\mathcal{S}$  has integral rank (respectively  $\mathcal{S}$  is represented by a vector bundle), and an orbifold is strongly Gorenstein if the inertial pair corresponding to the orbifold product is strongly Gorenstein.

We prove that the main results listed above actually hold for many inertial pairs. More precisely, we prove that a Gorenstein inertial pair determines a theory of Chern classes, and a strongly Gorenstein inertial pair also determines inertial K-theory operations ( $\lambda$ , Adams'  $\psi$ , and Grothendieck's  $\gamma$ ) compatible with the Chern classes. Moreover, when  $\mathcal{X} = [X/G]$  with  $G$  diagonalizable, then the K-theory operations determine a rationally augmented  $\lambda$ -ring structure on  $K(\mathcal{X})_{\mathbb{Q}}$  with an inertial dual which is a ring homomorphism, compatible with the Adams' operations and the inertial augmentation.

Since the inertial pair associated to the virtual orbifold product is always strongly Gorenstein [EJK12], we obtain the following result:

**Corollary.**

- (a) *The virtual orbifold product on  $K(I\mathcal{X})$  admits a Chern series homomorphism  $\tilde{c}_t : K(I\mathcal{X}) \rightarrow A^*(I\mathcal{X})_{\mathbb{Q}}[[t]]$  as well as compatible Adams  $\psi$  and  $\lambda$  operations on  $K(I\mathcal{X})_{\mathbb{Q}}$ .*
- (b) *If  $\mathcal{X} = [X/G]$  with  $G$  diagonalizable, then the virtual orbifold  $\lambda$  operations make  $K(I\mathcal{X})_{\mathbb{Q}}$  with its orbifold product into a rationally augmented  $\lambda$ -ring with a compatible inertial dual.*

This  $\lambda$ -ring structure has important consequences, as we now explain. Every  $\lambda$ -ring has a semigroup of so-called  $\lambda$ -positive elements which is an invariant of its  $\lambda$ -ring structure. In classical  $K$ -theory, where the  $\lambda$ -ring structure is obtained by taking exterior powers, the class of every vector bundle is  $\lambda$ -positive, and any  $\lambda$ -positive element shares many of the attributes of the class of a vector bundle (Proposition 7.1). Specifically, a  $\lambda$ -positive element in  $K$ -theory has an Euler class in both  $K$ -theory and Chow theory, and elements of  $\lambda$ -degree 1 behave formally like line bundles. Consequently, whenever an inertial  $K$ -theory ring has a  $\lambda$ -ring structure compatible with its inertial Chern classes and inertial Chern character, then its semigroup of  $\lambda$ -positive elements will satisfy the aforementioned properties, but where all products, rank, Chern classes, and the Chern character are the inertial ones. Furthermore, in many cases, the semigroup of  $\lambda$ -positive elements in inertial  $K$ -theory can be used to construct a nice presentation for inertial  $K$ -theory and inertial Chow theory.

A major motivation for the work in this paper is mirror symmetry. Ruan's (cohomological) Hyper-Kähler Resolution Conjecture (HKRC) predicts that for an orbifold  $\mathcal{X}$  with a hyper-Kähler resolution  $Z$ , the orbifold cohomology of  $\mathcal{X}$  should be isomorphic as a ring (after tensoring with  $\mathbb{C}$ ) to the usual cohomology of  $Z$ . In view of Ruan's conjecture, a natural problem is to investigate whether there is an orbifold  $\lambda$ -ring structure on orbifold  $K$ -theory that is isomorphic to the usual  $\lambda$ -ring structure on  $K(Z)$ .

One place to look for hyper-Kähler structures is the cotangent bundles of complex manifolds. These naturally carry a holomorphic symplectic structure, and in many cases these are hyper-Kähler. In [EJK12] we prove that if  $\mathcal{X} = [X/G]$ , then the virtual orbifold Chow ring of  $I\mathcal{X}$  (as defined in [GLS<sup>+</sup>07]) is isomorphic (after tensoring with  $\mathbb{C}$ ) to orbifold Chow ring of  $T^*I\mathcal{X}$ . Since the inertial pair defining the virtual orbifold product is strongly Gorenstein, we expect that the  $\lambda$ -ring structure (defined when  $G$  is diagonalizable) on  $K(I\mathcal{X})$  should be related to the usual  $\lambda$ -ring structure on  $K(Z)$ .

When  $\mathcal{X}$  is an orbifold,  $K(I\mathcal{X})$  typically has larger rank as an Abelian group than the corresponding Chow group  $A^*(I\mathcal{X})$ , while  $K(Z)$  and  $A^*(Z)$  have the same rank by the Riemann-Roch theorem for varieties. Thus, it is not reasonable to expect an isomorphism of  $\lambda$ -rings between  $K(I\mathcal{X})$  (with the virtual product) and  $K(Z)$  (with the tensor product).

However, the Riemann-Roch theorem for Deligne-Mumford stacks implies that a summand  $\hat{K}(I\mathcal{X})$ , corresponding to the completion at the classical augmentation ideal in  $K(I\mathcal{X})_{\mathbb{Q}}$ , is isomorphic as an Abelian group to  $A^*(I\mathcal{X})_{\mathbb{Q}}$ . Here we prove the remarkable result (Theorem 4.3) that if  $(\mathcal{X}, \mathcal{S})$  is any inertial pair, then the classical augmentation ideal in  $K(I\mathcal{X})_{\mathbb{Q}}$  and inertial augmentation ideal generate the same topology on the abelian group  $K(I\mathcal{X})$ . It follows that the summand  $\hat{K}(I\mathcal{X})$  inherits any inertial  $\lambda$ -ring structure from  $K(I\mathcal{X})$ .

This allows us to formulate  $\lambda$ -ring variant of the HKRC for orbifolds  $\mathcal{X} = [X/G]$  with  $G$  diagonalizable. Precisely, we expect there to be an isomorphism of  $\lambda$ -rings (after tensoring with  $\mathbb{C}$ ) between  $\widehat{K}(I\mathcal{X})$  with its virtual orbifold product and  $K(Z)$  where  $Z$  is a hyper-Kähler resolution of  $\mathbb{T}^*\mathcal{X}$ .

In the final section of the paper, we show that this conjecture holds in the case of the weighted projective line  $\mathbb{P}(1, n)$  for  $n = 2, 3$ . To do this we compute the  $\lambda$ -ring structure on  $K(I\mathbb{P}(1, n))_{\mathbb{C}}$  with the virtual orbifold product  $\star_{virt}$ , and the semi-group of  $\lambda$ -positive elements. Using the virtual orbifold line elements in this semi-group, we obtain a simple presentation of the ring  $(K(I\mathbb{P}(1, n))_{\mathbb{C}}, \star_{virt})$  and a simple description of the virtual first Chern classes. We then show that the augmentation completion  $\widehat{K}(I\mathbb{P}(1, n))_{\mathbb{C}}$  of  $K(I\mathbb{P}(1, n))_{\mathbb{C}}$  is isomorphic as a  $\lambda$ -ring to the ordinary K-theory  $K(Z)_{\mathbb{C}}$  of a toric resolution  $Z$  of the moduli space of the cotangent bundle stack  $\mathbb{T}^*\mathbb{P}(1, n)$ , thereby verifying the  $\lambda$ -ring version of the HKRC for these orbifolds. We also obtain an isomorphism of Chow rings  $(A^*(I\mathbb{P}(1, n))_{\mathbb{C}}, \star_{virt}) \cong A^*(Z)_{\mathbb{C}}$  that commutes with the corresponding Chern characters.

Furthermore, we show that the semigroup of inertial  $\lambda$ -positive elements induces an exotic integral lattice structure on  $(K(I\mathbb{P}(1, n))_{\mathbb{C}}, \star_{virt})$  (and  $(A_G^*(I_G X)_{\mathbb{C}}, \star_{virt})$ ) which corresponds to the ordinary integral lattice in  $K(Z)_{\mathbb{C}}$  (and  $A^*(Z)_{\mathbb{C}}$ , respectively).

Finally, our analysis suggests the following interesting question.

**Question 1.** *Does there exist a category associated to the crepant resolution  $Z$  whose Grothendieck group (with  $\mathbb{C}$ -coefficients) is isomorphic as a  $\lambda$ -ring to the virtual orbifold K-theory  $(K_G(I_G X)_{\mathbb{C}}, \star_{virt})$ ?*

**1.1. Outline of the paper.** We begin the paper by briefly reviewing the results of [EJK10, EJK12] on inertial pairs, inertial products, and inertial Chern characters.

We then briefly review  $\lambda$ -ring and  $\psi$ -ring structures in ordinary equivariant K-theory, including the Adams (power) operations, Bott classes, Grothendieck's  $\gamma$ -classes, and some relations among these and the Chern classes.

We then define, for Gorenstein inertial pairs, a theory of Chern classes and, for strongly Gorenstein inertial pairs, power (Adams) operations on inertial K-theory. Since the inertial pair associated to the virtual product of [GLS<sup>+</sup>07] is always strongly Gorenstein, this produces Chern classes and power operations in that theory.

We show that for strongly Gorenstein inertial pairs, inertial Chern classes satisfy a relation analogous to the that for usual Chern classes, expressing the Chern classes in terms of the orbifold  $\psi$  and  $\lambda$  operations. Finally we prove that if  $G$  is diagonalizable, the orbifold Adams operations are homomorphisms relative to the inertial product. In particular, this shows that the virtual K-theory of a toric Deligne-Mumford stack has  $\psi$ - and  $\lambda$ -ring structures. We also give an example to show that the diagonalizability condition is necessary for obtaining a  $\lambda$ -ring structure.

We then develop the theory of  $\lambda$ -positive elements for a  $\lambda$ -ring and show that  $\lambda$ -positive elements of degree  $d$  share many of the same properties as classes of rank  $d$ -vector bundles; for example, they have a top Chern class in Chow theory and an Euler class in K-theory. We also introduce the notion of an inertial dual which is needed to define the Euler class in inertial K-theory.

We conclude by working through some examples, including that of  $\mathcal{B}\mu_2$ , and the virtual K-theory of the weighted projective lines  $\mathbb{P}(1, 2)$  and  $\mathbb{P}(1, 3)$ .

The  $\lambda$ -positive elements, and especially the  $\lambda$ -line elements in the virtual theory, allow us to give a simple presentation of the K-theory ring with the virtual product and a simple description of the virtual first Chern classes. This allows us to prove that the completion of this ring with respect to the augmentation ideal is isomorphic as a  $\lambda$ -ring to the usual K-theory of the resolution of singularities of the cotangent orbifolds  $T^*\mathbb{P}(1, 2)$  and  $T^*\mathbb{P}(1, 3)$ , respectively.

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## 2. BACKGROUND MATERIAL

To make this paper self-contained, we recall some background material from the papers [EJK10, EJK12], but first we establish some notation and conventions.

**2.1. Notation.** We work entirely in the complex algebraic category. We will work exclusively with smooth Deligne-Mumford stacks  $\mathcal{X}$  which have *finite stabilizer*, by which we mean the inertia map,  $I\mathcal{X} \rightarrow \mathcal{X}$  is finite. We will also assume that every stack  $\mathcal{X}$  has the *resolution property*. This means that every coherent sheaf is the quotient of a locally free sheaf. This assumption has two consequences. The first is that the natural map  $K(\mathcal{X}) \rightarrow G(\mathcal{X})$  is an isomorphism, where  $K(\mathcal{X})$  is the Grothendieck ring of vector bundles, and  $G(\mathcal{X})$  is the Grothendieck group of coherent sheaves. The second consequence is that  $\mathcal{X}$  is a *quotient stack* [EHKV01]. This means that  $\mathcal{X} = [X/G]$ , where  $G$  is a linear algebraic group acting on a scheme or algebraic space  $X$ .

If  $\mathcal{X}$  is a smooth Deligne-Mumford stack, we will implicitly choose a presentation  $\mathcal{X} = [X/G]$ . This allows us to identify the Grothendieck ring  $K(\mathcal{X})$  with the equivariant Grothendieck ring  $K_G(X)$ , and the Chow ring  $A^*(\mathcal{X})$  with the equivariant Chow ring  $A_G^*(X)$ . We will use the notation  $K(\mathcal{X})$  and  $K_G(X)$  (respectively  $A^*(\mathcal{X})$  and  $A_G^*(X)$ ) interchangeably.

**Definition 2.1.** Let  $G$  be an algebraic group acting on a scheme or algebraic space  $X$ . We define the *inertia space*

$$I_G X := \{(g, x) | gx = x\} \subseteq G \times X.$$

There is an induced action of  $G$  on  $I_G X$  given by  $g \cdot (m, x) = (gmg^{-1}, gx)$ . The quotient stack  $I\mathcal{X} := [I_G X/G]$  is the inertia stack of the quotient stack  $\mathcal{X} := [X/G]$ .

More generally, we define the higher inertia spaces to be the  $k$ -fold fiber products

$$\mathbb{I}_G^k X = I_G X \times_X \dots \times_X I_G X.$$

The quotient stack  $\mathbb{I}^k \mathcal{X} := [\mathbb{I}_G^k X/G]$  is the corresponding higher inertia stack.

The assumption that  $\mathcal{X}$  has finite stabilizer means that the projection  $I_G X \rightarrow X$  is a finite morphism. The composition  $\mu: G \times G \longrightarrow G$  induces a composition  $\mu: I_G X \longrightarrow X$ . This composition makes  $I_G X$  into an  $X$ -group with identity section  $X \longrightarrow I_G X$  given by  $x \mapsto (1, x)$ .

**Definition 2.2.** Let  $\Psi \subset G$  be a conjugacy class and let  $I(\Psi) = \{(g, x) | gx = x, g \in \Psi\} \subset G \times X$ . More generally, if  $\Phi \subset G^\ell$  is a diagonal conjugacy class, we define  $\mathbb{I}^\ell(\Phi) = \{(m_1, \dots, m_\ell, x) | (m_1, \dots, m_\ell) \in \Phi \text{ and } m_i x = x \text{ for all } i = 1, \dots, \ell\}$ .

By definition,  $I(\Psi)$  and  $\mathbb{I}^\ell(\Phi)$  are  $G$ -invariant subsets of  $I_G X$  and  $\mathbb{I}_G^\ell(X)$ , respectively. If  $G$  acts with finite stabilizer on  $X$ , then  $I(\Psi)$  is empty unless  $\Psi$  consists of elements of finite order. Likewise,  $\mathbb{I}^\ell(\Phi)$  is empty unless every  $\ell$ -tuple  $(m_1, \dots, m_\ell) \in \Phi$  generates a finite group. Since conjugacy classes of elements of finite order are closed,  $I(\Psi)$  and  $\mathbb{I}^\ell(\Phi)$  are closed.

**Proposition 2.3.** ([EJK10, Prop. 2.11, 2.17]) *If  $G$  acts with finite stabilizer on  $X$ , then  $I(\Psi) = \emptyset$  for all but finitely many  $\Psi$ , and the  $I(\Psi)$  are unions of connected components of  $I_G X$ . Likewise,  $\mathbb{I}^\ell(\Phi)$  is empty for all but finitely many diagonal conjugacy classes  $\Phi \subset G^\ell$ , and each  $\mathbb{I}^\ell(\Phi)$  is a union of connected components of  $\mathbb{I}_G^\ell(X)$ .*

We frequently work with a group  $G$  acting on a space  $X$  where the quotient stack  $[X/G]$  is not connected. As a consequence, some care is required in the definition of the rank and Euler class of a vector bundle. Note that for any  $X$ , the group  $A_G^0(X)$  satisfies  $A_G^0(X) = \mathbb{Z}^\ell$ , where  $\ell$  is the number of connected components of the quotient stack  $\mathcal{X} = [X/G]$ .

**Definition 2.4.** For any  $\alpha \in K_G(X)$  we define the *rank* of  $\alpha$  to be  $\text{rk}(\alpha) := \text{Ch}^0(\alpha) \in A_G^0(X) = \mathbb{Z}^\ell$ .

If  $E$  is a  $G$  equivariant vector bundle on  $X$ , then the rank of  $E$  on the connected components of  $\mathcal{X} = [X/G]$  is bounded, since we assume that  $\mathcal{X}$  has finite type.

**Definition 2.5.** If  $E$  is a  $G$  equivariant vector bundle on  $X$ , the element  $\lambda_{-1}(E^*) = \sum_{i=0}^\infty (-1)^i [\Lambda^i E^*] \in K_G(X)$  is called the *K-theoretic Euler class* of  $E$ . (Note that this sum is finite.)

Likewise, we define the element  $c_{\text{top}}(E) \in A_G^*(X)$ , corresponding to the sum of the top Chern classes of  $E$  on each connected component of  $[X/G]$ , to be the *Chow-theoretic Euler class* of  $E$ . These definitions can be extended to any non-negative element by multiplicativity. It will be convenient to use the symbol  $\text{eu}(\mathcal{F})$  to denote both of these Euler classes for a non-negative element  $\mathcal{F} \in K_G(X)$ .

**Definition 2.6.** We define the *augmentation homomorphism*  $\epsilon: K_G(Y) \longrightarrow K_G(Y)$  to be the map which, for each connected component  $[U/G]$  of  $[Y/G]$ , sends each  $\mathcal{F}$  in  $K_G(Y)$  supported on  $U$  to the rank of  $\mathcal{F}$  times the structure sheaf  $\mathcal{O}_U$

$$\epsilon(\mathcal{F}|_U) := \text{Ch}^0(\mathcal{F}|_U) \mathcal{O}_U.$$

It follows immediately that

$$\epsilon \circ \epsilon = \epsilon, \tag{1}$$

$$\text{Ch}^0 \circ \epsilon = \text{Ch}^0. \tag{2}$$

**Definition 2.7.** An *augmented ring* is a commutative unital ring  $(R, \cdot, 1)$  together with an endomorphism  $\epsilon: R \longrightarrow R$  satisfying Equation (1). The kernel of  $\epsilon$  is called the *augmentation ideal*.

**2.2. Inertial products, Chern characters, and inertial pairs.** We review here the results from [EJK12], defining a generalization of orbifold cohomology, obstruction bundles, age grading, and stringy Chern character, by defining *inertial products* on  $K_G(I_G X)$  and  $A_G^*(I_G X)$  using *inertial pairs*  $(\mathcal{R}, \mathcal{S})$ , where  $\mathcal{R}$  is a  $G$ -equivariant vector bundle on  $\mathbb{I}_G^2 X$  and  $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$  is a non-negative class satisfying certain compatibility properties.

For each such pair, there is also a rational grading in Chow, and a Chern character ring homomorphism. There are many inertial pairs, and hence there are many associative inertial products on  $K_G(I_G X)$  and  $A_G^*(I_G X)$  with rational gradings and Chern character ring homomorphisms. The orbifold products on  $K(I\mathcal{X})$  and  $A^*(I\mathcal{X})$  and the Chern character homomorphism of [JKK07] are a special case, as is the virtual product of [GLS<sup>+</sup>07].

**Definition 2.8.** If  $\mathcal{R}$  is a vector bundle on  $\mathbb{I}_G^2 X$  we define products on  $A_G^*(I_G X)$  and  $K_G(I_G X)$  via the following formula:

$$x \star_{\mathcal{R}} y := \mu_* (e_1^* x \cdot e_2^* y \cdot \text{eu}(\mathcal{R})), \quad (3)$$

where  $x, y \in A_G^*(I_G X)$  (respectively  $K_G(I_G X)$ ) where  $\mu: \mathbb{I}_G^2 X \rightarrow I_G X$  is the composition map and  $e_1, e_2: \mathbb{I}_G^2 X \rightarrow I_G X$  are the evaluation maps.

To define an inertial pair requires a little more notation from [EJK10], which we recall here. Consider  $(m_1, m_2, m_3) \in G^3$  such that  $m_1 m_2 m_3 = 1$ , and let  $\Phi_{1,2,3}$  be the conjugacy class of  $(m_1, m_2, m_3)$ . Let  $\Phi_{12,3}$  be the conjugacy class of  $(m_1 m_2, m_3)$  and  $\Phi_{1,23}$  the conjugacy class of  $(m_1, m_2 m_3)$ . Let  $\Phi_{i,j}$  be the conjugacy class of the pair  $(m_i, m_j)$  with  $i < j$ . Finally, let  $\Psi_{123}$  be the conjugacy class of  $m_1 m_2 m_3$ ; let  $\Psi_{ij}$  be the conjugacy class of  $m_i m_j$ ; and let  $\Psi_i$  be the conjugacy class of  $m_i$ . There are composition maps  $\mu_{12,3}: \mathbb{I}^3(\Phi_{1,2,3}) \rightarrow \mathbb{I}^2(\Phi_{12,3})$ , and  $\mu_{1,23}: \mathbb{I}^3(\Phi_{1,2,3}) \rightarrow \mathbb{I}^2(\Phi_{1,23})$ , and  $\mu_{123}: \mathbb{I}^3(\Phi_{1,2,3}) \rightarrow I(\Psi_{123})$ . The various maps we have defined are related by the following Cartesian diagrams of l.c.i. morphisms.

$$\begin{array}{ccc} \mathbb{I}^3(\Phi_{1,2,3}) & \xrightarrow{e_{1,2}} & \mathbb{I}^2(\Phi_{1,2}) \\ \mu_{12,3} \downarrow & & \downarrow \mu \\ \mathbb{I}^2(\Phi_{12,3}) & \xrightarrow{e_1} & I(\Psi_{12}) \end{array} \quad \begin{array}{ccc} \mathbb{I}^3(\Phi_{1,2,3}) & \xrightarrow{e_{2,3}} & \mathbb{I}^2(\Phi_{2,3}) \\ \mu_{1,23} \downarrow & & \downarrow \mu \\ \mathbb{I}^2(\Phi_{1,23}) & \xrightarrow{e_1} & I(\Psi_{23}) \end{array} \quad (4)$$

Let  $E_{1,2}$  and  $E_{2,3}$  be the respective excess normal bundles of the two diagrams (4).

**Definition 2.9.** Given a non-negative element  $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$  and  $G$ -equivariant vector bundle  $\mathcal{R}$  on  $\mathbb{I}_G^2 X$  we say that  $(\mathcal{R}, \mathcal{S})$  is an *inertial pair* if the following conditions hold:

- (a) The identity  $\mathcal{R} = e_1^* \mathcal{S} + e_2^* \mathcal{S} - \mu^* \mathcal{S} + T_{\mu}$  holds in  $K_G(\mathbb{I}_G^2 X)$
- (b)  $\mathcal{R}|_{\mathbb{I}^2(\Phi)} = \mathcal{O}$  for every conjugacy class  $\Phi \subset G \times G$  such that  $e_1(\Phi) = 1$  or  $e_2(\Phi) = 1$ .
- (c)  $i^* \mathcal{R} = \mathcal{R}$ , where  $i: \mathbb{I}_G^2 X \rightarrow \mathbb{I}_G^2 X$  is the isomorphism  $i(m_1, m_2, x) = (m_1 m_2 m_1^{-1}, m_1, x)$ .
- (d)  $e_{1,2}^* \mathcal{R} + \mu_{12,3}^* \mathcal{R} + E_{1,2} = e_{2,3}^* \mathcal{R} + \mu_{1,23}^* \mathcal{R} + E_{2,3}$  for each triple  $m_1, m_2, m_3$  with  $m_1 m_2 m_3 = 1$ .

**Proposition 2.10** ([EJK10, §3]). *If  $(\mathcal{R}, \mathcal{S})$  is an inertial pair, then the  $\star_{\mathcal{R}}$  product is commutative and associative with identity  $\mathbf{1}_X$ , where  $\mathbf{1}_X$  is the identity class*

in  $A_G^*(X)$  (respectively  $K_G(X)$ ), viewed as a summand in  $A_G^*(I_G X)$  (respectively  $K_G(I_G X)$ ).

**Proposition 2.11.** [EJK12, Prop 3.8] *If  $(\mathcal{R}, \mathcal{S})$  is an inertial pair, then the map*

$$\widetilde{\mathcal{C}h}: K_G(I_G X)_{\mathbb{Q}} \longrightarrow A_G^*(I_G X)_{\mathbb{Q}},$$

*defined by*

$$\widetilde{\mathcal{C}h}(V) = \text{Ch}(V) \cdot \text{Td}(-\mathcal{S}),$$

*is a ring homomorphism with respect to the  $\star_{\mathcal{R}}$ -inertial products on  $K_G(I_G X)$  and  $A_G^*(I_G X)$ .*

**Definition 2.12.** We define the  $\mathcal{S}$ -age on a component  $U$  of  $I_G X$  corresponding to a connected component  $[U/G]$  of  $I_G X$  to be the rational rank of  $\mathcal{S}$  on the component  $U$ :

$$\text{age}_{\mathcal{S}}(U) = \text{rk}(\mathcal{S})_U.$$

We define the  $\mathcal{S}$ -degree of an element  $x \in A_G^*(I_G X)$  on such a component  $U$  of  $I_G X$  to be

$$\deg_{\mathcal{S}} x|_U = \deg x|_U + \text{age}_{\mathcal{S}}(U),$$

where  $\deg x$  is the degree with respect to the usual grading by codimension on  $A_G^*(I_G X)$ . Similarly, let  $\mathcal{F}$  in  $K_G(I_G X)$  be an element supported on  $U$ , then its  $\mathcal{S}$ -degree is

$$\deg_{\mathcal{S}} \mathcal{F} = \text{age}_{\mathcal{S}}(U) \bmod \mathbb{Z}.$$

This yields a  $\mathbb{Q}/\mathbb{Z}$ -grading of the group  $K_G(I_G X)$ .

**Proposition 2.13.** [EJK12, Prop 3.11] *If  $(\mathcal{R}, \mathcal{S})$  is an inertial pair, then the  $\mathcal{R}$ -inertial products on  $A_G^*(I_G X)$  and  $K_G(I_G X)$  respect the  $\mathcal{S}$ -degrees. Furthermore, the inertial Chern character homomorphism  $\widetilde{\mathcal{C}h}: K_G(I_G X) \longrightarrow A_G^*(I_G X)$  preserves the  $\mathcal{S}$ -degree modulo  $\mathbb{Z}$ .*

**Definition 2.14.** Let  $A_G^{\{q\}}(I_G X)$  be the subspace in  $A_G^*(I_G X)$  of elements with an  $\mathcal{S}$ -degree of  $q \in \mathbb{Q}^{\ell}$ , where  $\ell$  is the number of connected components of  $I\mathcal{X}$ .

**Definition 2.15.** Let  $(\mathcal{R}, \mathcal{S})$  be an inertial pair, and let  $\ell$  be the number of connected components of  $I\mathcal{X} = [I_G X/G]$ . The subring of  $K_G(I_G X)$  consisting of all elements of  $\mathcal{S}$ -grading  $0 \in (\mathbb{Q}/\mathbb{Z})^{\ell}$  is called the *Gorenstein subring*  $\check{K}_G(I_G X)$  of  $K_G(I_G X)$ , and the subring of  $A_G^*(I_G X)$  consisting of all elements of  $\mathcal{S}$ -degree in  $\mathbb{Z}^{\ell} \subseteq \mathbb{Q}^{\ell}$  is called the *Gorenstein subring*  $\check{A}_G(I_G X)$  of  $A_G^*(I_G X)$ .

**Definition 2.16.** Given a class  $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$ , the restricted homomorphism  $\widetilde{\mathcal{C}h}^0: K_G(I_G X) \longrightarrow A_G^{\{0\}}(I_G X)$  is called the *inertial rank for  $\mathcal{S}$*  or just the  $\mathcal{S}$ -rank.

The *inertial augmentation homomorphism*  $\tilde{\epsilon}: K_G(I_G X) \longrightarrow K_G(I_G X)$  is the map which for each connected component  $[U/G]$  of  $[(I_G X)/G]$  sends each  $\mathcal{F}$  in  $K_G(I_G X)$  supported on  $U$  to

$$\tilde{\epsilon}(\mathcal{F}|_U) = \widetilde{\mathcal{C}h}^0(\mathcal{F}|_U)\theta_U.$$

Hence, under the conditions of the previous Proposition,  $(K_G(I_G X), \star, 1, \tilde{\epsilon})$  is an augmented ring.

**Remark 2.17.** Note that the restriction  $\widetilde{\mathcal{E}h}^0(\mathcal{F})|_U$  of the inertial rank to a given component is equal to the classical rank if the  $\mathcal{S}$ -age of that component is zero, and  $\widetilde{\mathcal{E}h}^0(\mathcal{F})|_U$  vanishes if the age is non-zero.

**Definition 2.18.** An inertial pair  $(\mathcal{R}, \mathcal{S})$  is called *Gorenstein* if  $\mathcal{S}$  has integral rank and *strongly Gorenstein* if  $\mathcal{S}$  is represented by a vector bundle.

The Deligne-Mumford stack  $\mathcal{X} = [X/G]$  is *strongly Gorenstein* if the inertial pair  $(\mathcal{R} = LR(\mathbb{T}), \mathcal{S})$  associated to the orbifold product (as in [EJK10]) is strongly Gorenstein.

**2.3. Inertial pairs associated to vector bundles.** As mentioned above, for each vector bundle  $V \in K_G(X)$ , we can construct two inertial pairs  $(\mathcal{R}^+V, \mathcal{S}^+V)$  and  $(\mathcal{R}^-V, \mathcal{S}^-V)$ .

**Definition 2.19** ([EJK12]). Let  $\Phi \subset G \times G$  be a diagonal conjugacy class. We may identify  $K_G(\mathbb{I}^2(\Phi))$  with  $K_{Z_G(g_1, g_2)}(X^{g_1, g_2})$  for any  $(g_1, g_2) \in \Phi$ , and we define elements  $R^+V$  and  $R^-V$  in  $K_G(\mathbb{I}_G^2 X)$  by setting the component of  $R^+V$  in  $K_G(\mathbb{I}^2(\Phi))$  to be

$$R^+V|_{\mathbb{I}^2(\Phi)} = L(g_1)(V|_{X^{g_1, g_2}}) + L(g_2)(V|_{X^{g_1, g_2}}) - L(g_1 g_2)(V|_{X^{g_1, g_2}}), \quad (5)$$

and the component of  $R^-V$  in  $K_G(\mathbb{I}^2(\Phi))$  to be

$$R^-V|_{\mathbb{I}^2(\Phi)} = L(g_1^{-1})(V|_{X^{g_1, g_2}}) + L(g_2^{-1})(V|_{X^{g_1, g_2}}) - L(g_2^{-1} g_1^{-1})(V|_{X^{g_1, g_2}}), \quad (6)$$

where  $L(g)(E)$  is the logarithmic trace of  $E$ , as defined in [EJK10, Def 4.3]. This definition is independent of the choice of the pair  $(g_1, g_2) \in \Phi$ .

Similarly, we define classes  $S^\pm V \in K_G(I_G X)_{\mathbb{Q}}$  by setting the restriction of  $S^\pm V$  to the summand  $K_G(I(\Psi))$  of  $K_G(I_G X)$  to be the class Morita equivalent to  $L(g^{\pm 1})(V) \in K_{Z_G(g)}(X^g)$ , where  $g \in \Psi$  is any element of the conjugacy class  $\Psi \subset G$ .

**Theorem 2.20** ([EJK12]). *For any  $G$ -equivariant vector bundle  $V$  on  $X$ , let  $\mathbb{T}$  denote the tangent bundle of  $\mathcal{X} = [X/G]$ . The pairs  $(\mathcal{R}^+V, \mathcal{S}^+V) := (LR(\mathbb{T}) + R^+V, \mathcal{S}\mathbb{T} + S^+V)$  and  $(\mathcal{R}^-V, \mathcal{S}^-V) := (LR(\mathbb{T}) + R^-V, \mathcal{S}\mathbb{T} + S^-V)$  are inertial pairs; hence, they define commutative, associative inertial products with a Chern character homomorphism.*

The orbifold products on  $K(I\mathcal{X})$  and  $A^*(I\mathcal{X})$  and the Chern character homomorphism of [JKK07] correspond to the inertial pair arising from the trivial vector bundle on  $\mathcal{X}$ , and the virtual product considered by [GLS<sup>+</sup>07] is the product associated to the inertial pair  $(\mathcal{R}^-T, \mathcal{S}^-T)$ .

### 3. REVIEW OF $\lambda$ -RING AND $\psi$ -RING STRUCTURES IN EQUIVARIANT K-THEORY

In this section, we review the  $\lambda$ - and  $\psi$ -ring structures in equivariant K-theory and describe the Bott cannibalistic classes  $\theta^i$ , as well as the Grothendieck  $\gamma$ -classes. The main theorems about these classes are the Adams-Riemann-Roch Theorem (Theorem 3.18) and Theorem 3.13, which describes relations among the Chern Character, the  $\psi$ -classes, the Chern classes, and the  $\gamma$ -classes.

Recall that a  $\lambda$ -ring is a commutative ring  $R$  with unity 1 and with a map  $\lambda_t : R \longrightarrow R[[t]]$ , where

$$\lambda_t(a) =: \sum_{i \geq 0} \lambda^i(a) t^i, \quad (7)$$

such that the following are satisfied for all  $x, y$  in  $R$  and for all integers  $m, n \geq 0$ :

- (1)  $\lambda^0(a) = 1$  for all  $a \in R$ ,
- (2)  $\lambda_t(1) = 1 + t$ ,
- (3)  $\lambda^1(x) = x$ ,
- (4)  $\lambda_t(x + y) = \lambda_t(x)\lambda_t(y)$ ,
- (5)  $\lambda^n(xy) = \mathbf{P}_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y))$ ,
- (6)  $\lambda^m(\lambda^n(x)) = \mathbf{P}_{m,n}(\lambda^1(x), \dots, \lambda^{mn}(x))$ ,

where  $\mathbf{P}_n$ , and  $\mathbf{P}_{m,n}$  are certain universal polynomials which are independent of  $x$  and  $y$  (see [FL85, §I.1]).

If, moreover, the  $\lambda$ -ring  $R$  is a  $\mathbb{K}$ -algebra, where  $\mathbb{K}$  is a field of characteristic 0, then we call  $(R, \cdot, 1, \lambda)$  a  $\lambda$ -algebra over  $\mathbb{K}$  if, for all  $\alpha$  in  $\mathbb{K}$  and all  $a$  in  $R$ , we have

$$\lambda_t(\alpha a) = \lambda_t(a)^\alpha := \exp(\alpha \log \lambda_t). \quad (8)$$

Note that  $\log \lambda_t$  makes sense because any series for  $\lambda_t$  starts with 1.

**Remark 3.1.** The significance of the universal polynomials in the definition of a  $\lambda$ -ring is that one can calculate  $\lambda^n(xy)$  and  $\lambda^m(\lambda^n(x))$  in terms of  $\lambda^i(x)$  and  $\lambda^j(y)$  by applying a formal splitting principle.

For example, suppose we wish to express  $\lambda_t(x \cdot y)$  in terms of  $\lambda_t(x)$  and  $\lambda_t(y)$ . First, replace  $x$  by the formal sum  $x \mapsto \sum_{i=1}^{\infty} x_i$ , where we assume that  $\lambda_t(x_i) = 1 + tx_i$  for all  $i$ , and similarly replace  $y$  by the formal sum  $y \mapsto \sum_{i=1}^{\infty} y_i$  in  $\lambda_t(x \cdot y)$ , where we assume that  $\lambda_t(y_i) = 1 + ty_i$  for all  $i$ . The fact that  $\lambda_t(x_i) = 1 + tx_i$  and  $\lambda_t(y_j) = 1 + ty_j$  means that  $\lambda_t(x_i y_j) = 1 + tx_i y_j$ , and multiplicativity gives us

$$\lambda_t(x \cdot y) \mapsto \prod_{i,j=1}^{\infty} (1 + tx_i y_j).$$

Therefore,  $\lambda^n(x \cdot y)$  corresponds to the  $n$ -th elementary symmetric function  $e_n(xy)$  in the variables  $\{x_i y_j\}_{i,j=1}^{\infty}$ , but  $e_n(xy)$  can be uniquely expressed as a polynomial  $\mathbf{P}_n$  in the variables  $\{e_1(x), \dots, e_n(x), e_1(y), \dots, e_n(y)\}$ , where  $e_q(x)$  denotes the  $q$ -th elementary symmetric function in the  $\{x_i\}_{i=1}^{\infty}$  variables and  $e_r(y)$  denotes the  $r$ -th elementary symmetric function in the  $\{y_i\}_{i=1}^{\infty}$  variables. Replacing  $e_q(x)$  by  $\lambda^q(x)$  and  $e_r(y)$  by  $\lambda^r(y)$  in  $\mathbf{P}_n$  for all  $q, r \in \{1, \dots, n\}$  yields the universal polynomial  $\mathbf{P}_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y))$  appearing in the definition of a  $\lambda$ -ring.

A similar analysis holds for  $\mathbf{P}_{m,n}$ .

A closely related structure is that of a  $\psi$ -ring.

**Definition 3.2.** A commutative ring  $R$  with unity 1, together with a collection of maps  $\psi^n : R \longrightarrow R$  for each  $n \geq 1$ , is called a  $\psi$ -ring if, for all  $x, y$  in  $R$  and for all integers  $n \geq 1$ , we have

- (1)  $\psi^1(x) = x$ ,
- (2)  $\psi^n(x + y) = \psi^n(x) + \psi^n(y)$ ,
- (3)  $\psi^n(xy) = \psi^n(x)\psi^n(y)$ ,
- (4)  $\psi^m(\psi^n(x)) = \psi^{mn}(x)$ .

The map  $\psi^i : R \longrightarrow R$  is called the  $i$ -th Adams operation (or power operation).

If the  $\psi$ -ring  $(R, \cdot, 1, \psi)$  is a  $\mathbb{K}$ -algebra, then  $(R, \cdot, 1, \psi)$  is said to be a  $\psi$ -algebra over  $\mathbb{K}$  if, in addition,  $\psi^n$  is a  $\mathbb{K}$ -linear map.

**Theorem 3.3** (cf. [Knu73] p.49). *Let  $(R, \cdot, 1, \lambda)$  be a commutative  $\lambda$ -ring, and let  $\psi_t : R \longrightarrow R[[t]]$  be given by*

$$\psi_t = -t \frac{d \log \lambda_{-t}}{dt}. \quad (9)$$

*Expanding  $\psi_t$  as  $\psi_t := \sum_{n \geq 1} (-1)^n \psi^n t^n$  defines  $\psi^n : R \longrightarrow R$  for all  $n \geq 1$ , and the resulting ring  $(R, \cdot, 1, \psi)$  is a  $\psi$ -ring.*

*Conversely, if  $(R, \cdot, 1, \psi)$  is a  $\psi$ -ring and if  $\lambda_t : R_{\mathbb{Q}} \longrightarrow R_{\mathbb{Q}}[[t]]$  is defined by*

$$\lambda_t = \exp \left( \sum_{r \geq 1} (-1)^{r-1} \psi^r \frac{t^r}{r} \right), \quad (10)$$

*then  $(R_{\mathbb{Q}}, \cdot, 1, \lambda)$  is a  $\lambda$ -algebra over  $\mathbb{Q}$ .*

**Remark 3.4.** As in Remark 3.1, the  $k$ -th  $\lambda$  operation,  $\lambda^k$ , corresponds to the  $k$ -th elementary symmetric function. Equation (10) implies that the  $k$ -th power operation,  $\psi^k$ , corresponds to the  $k$ -th power sum symmetric function since this equation is nothing more than the relationship between the elementary symmetric functions and the power sums.

Let  $G$  be an algebraic group acting on an algebraic space  $X$ . The Grothendieck ring  $(K_G(X), \cdot, 1)$  of  $G$ -equivariant vector bundles on  $X$  is a commutative ring with unity, where we have used  $\cdot$  to denote the tensor product and  $1$  to denote the structure sheaf  $\mathcal{O}_X$  of  $X$ .

It is well known that (non-equivariant) K-theory with exterior powers is a  $\lambda$  ring and the associated  $\psi$  ring satisfies  $\psi^k(\mathcal{L}) = \mathcal{L}^{\otimes k}$  for all line bundles  $\mathcal{L}$ . Since the exterior powers (and the associated  $\psi$  operations) respect  $G$ -equivariance, the following theorem is immediate.

**Theorem 3.5** (c.f. [Köc98], Lemma 2.4). *For any  $G$ -equivariant vector bundle  $V$  on  $X$ , define  $\lambda^k([V])$  to be the class  $[\Lambda^k(V)]$  of the  $k$ -th exterior power of  $V$ . This defines a  $\lambda$ -ring structure  $(K_G(X), \cdot, 1, \lambda)$  on all of  $K_G(X)$ . The corresponding  $\psi$ -ring structure on  $(K_G(X), \cdot, 1)$  has the property that for any line bundle  $\mathcal{L}$  and integer  $k \geq 1$ ,*

$$\psi^k(\mathcal{L}) = \mathcal{L}^{\otimes k}. \quad (11)$$

**Remark 3.6.** The splitting principle [FL85] guarantees that Equation (11) can be used to define the Adams operations on all of  $K_G(X)$ .

**Remark 3.7.** The  $\lambda$ -ring  $K_G(X)$  has still more structure, since any element can be represented as a difference of vector bundles. The collection of classes of vector bundles  $\mathbf{E}$  in  $K_G(X)$  endows the  $\lambda$ -ring  $K_G(X)$  with a *positive structure* [FL85]. Roughly speaking, this means that  $\mathbf{E}$  is a subset of the  $\lambda$ -ring consisting of elements of non-negative rank such that any element in the ring can be written as differences of elements in  $\mathbf{E}$ , and for any  $\mathcal{F}$  of rank- $d$  in  $\mathbf{E}$ ,  $\lambda_t(\mathcal{F})$  is a degree  $d$  polynomial in  $t$ , and  $\lambda^d(\mathcal{F})$  is invertible (i.e.,  $\lambda^d(\mathcal{F})$  is a line bundle). Furthermore,  $\mathbf{E}$  is closed under addition (but not subtraction) and multiplication;  $\mathbf{E}$  contains the non-negative integers; and there are special rank-one elements in  $\mathbf{E}$ , namely the line bundles; and various other properties also hold. A positive structure on a  $\lambda$ -ring, if it exists, need not be uniquely determined by the  $\lambda$ -ring structure, nor does a general  $\lambda$ -ring possess a positive structure. For example, if  $G = \mathrm{GL}_n$ , then the representation ring  $R(G)$  can be identified as a subring of Weyl group-invariant

elements in the representation ring  $R(T)$ , where  $T$  is a maximal torus and the  $\lambda$ -ring structure on  $R(T)$  restricts to the usual  $\lambda$ -ring structure on  $R(G)$ . However, the natural set of positive elements in  $R(T)$  is generated by the characters of  $T$ , and this restricts to the set of positive symmetric linear combinations of characters which contains, but does not equal, the set of irreducible representations of  $G$ .

However, in Section 6, we will introduce a different but related notion called a  $\lambda$ -positive structure, which is a natural invariant of a  $\lambda$ -ring. It is this notion which will play a central role in our analysis of inertial K-theory.

The  $\lambda$ - and  $\psi$ -ring structures behave nicely with respect to the augmentation.

**Proposition 3.8.** *For all  $\mathcal{F}$  in  $K_G(X)$  and integers  $n \geq 1$ , we have*

$$\epsilon(\psi^n(\mathcal{F})) = \psi^n(\epsilon(\mathcal{F})) = \epsilon(\mathcal{F}), \quad (12)$$

and

$$\epsilon(\lambda_t(\mathcal{F})) = \lambda_t(\epsilon(\mathcal{F})) = (1+t)^{\epsilon(\mathcal{F})}. \quad (13)$$

*Proof.* Assume that  $[X/G]$  is connected. Equation (13) holds if  $\mathcal{F}$  is a rank  $d$   $G$ -equivariant vector bundle on  $X$  since  $\lambda^i(\mathcal{F})$  has rank  $\binom{d}{i}$ . Since  $K_G(X)$  is generated under addition by isomorphism classes of vector bundles, the same equation holds for all  $\mathcal{F}$  in  $K_G(X)$  by multiplicativity of  $\lambda_t$ .

If  $[X/G]$  is not connected, then we have the ring isomorphism  $K_G(X) = \bigoplus_{\alpha} K_G(X_{\alpha})$ , where the sum is over  $\alpha$  such that  $[X_{\alpha}/G]$  is a connected component of  $[X/G]$ . Equation (13) follows from multiplicativity of  $\lambda_t$ . Equation (12) follows Equations (13) and (9).  $\square$

This motivates the following definition.

**Definition 3.9.** Let  $(R, \cdot, 1, \epsilon)$  be an augmented ring.  $(R, \cdot, 1, \psi, \epsilon)$  is said to be an *augmented  $\psi$ -ring* if  $(R, \cdot, 1, \psi)$  is a  $\psi$ -ring and Equation (12) holds for all integers  $n \geq 1$ . Let  $\psi^0 := \epsilon$ .

We say that  $(R, \cdot, 1, \lambda, \epsilon)$  is an *augmented  $\lambda$ -ring* if  $(R, \cdot, 1, \lambda)$  is a  $\lambda$ -ring which is a  $\mathbb{Q}$ -algebra, if  $\epsilon$  is an augmentation of  $R$ , and if Equation (13) holds. Here the expression  $(1+t)^x$  for an arbitrary element  $x$  of the  $\mathbb{Q}$ -algebra  $R$  should be taken to mean

$$(1+t)^x := \sum_{n=0}^{\infty} \binom{x}{n} t^n,$$

where

$$\binom{x}{n} := \frac{x^n}{n!} := \frac{\prod_{i=0}^{n-1} (x-i)}{n!}.$$

**Remark 3.10.** The definition  $\psi^0 = \epsilon$  is consistent with conditions 2–4 in the definition of a  $\psi$ -ring (Definition 3.2).

The previous proposition implies that ordinary equivariant K-theory is an augmented  $\psi$ -ring. In fact, the equivariant Chow ring is also an augmented  $\psi$ -ring.

**Definition 3.11.** For all  $n \geq 1$ , the homomorphism  $\psi^n : A_G^*(X) \longrightarrow A_G^*(X)$  defined by

$$\psi^n(v) = n^d v \quad (14)$$

for all  $v$  in  $A_G^d(X)$  endows  $A_G^*(X)$  with the structure of a  $\psi$ -ring and, therefore,  $A_G^*(X)_{\mathbb{Q}}$  with the structure of a  $\lambda$ -ring. The *augmentation*  $\epsilon : A_G^*(X) \longrightarrow A_G^0(X)$  is the canonical projection.

Associated to any  $\lambda$ -ring there is another (pre- $\lambda$ -ring) structure usually denoted by  $\gamma$ .

**Definition 3.12.** Let  $(R, \cdot, 1, \lambda)$  be a  $\lambda$ -ring. Define the *Grothendieck  $\gamma$ -classes*  $\gamma_t : R \longrightarrow R[[t]]$  by the formula

$$\gamma_t := \sum_{i=0}^{\infty} t^i \gamma^i := \lambda_{t/(1-t)}. \quad (15)$$

**Theorem 3.13** (cf. [FL85]). *If  $Y$  is a connected algebraic space with a proper action of a linear algebraic group  $G$ , and if, for each non-negative integer  $i$ , we denote by  $\text{Ch}^i$  the degree- $i$  part of the Chern character and by  $c^i$  the  $i$ th Chern class, then the following equations hold for all integers  $n \geq 1$ ,  $i \geq 0$  and for all  $\mathcal{F}$  in  $K_G(Y)$ :*

$$\text{Ch}^i \circ \psi^n = n^i \text{Ch}^i, \quad (16)$$

$$c_t(\mathcal{F}) = \exp \left( \sum_{n \geq 1} (-1)^{n-1} (n-1)! t^n \text{Ch}^n(\mathcal{F}) \right), \quad (17)$$

and

$$c^i(\mathcal{F}) = \text{Ch}^i(\gamma^i(\mathcal{F} - \epsilon(\mathcal{F}))). \quad (18)$$

**Remark 3.14.** Equation (16) is precisely the statement that the Chern character  $\text{Ch} : K_G(X)_{\mathbb{Q}} \longrightarrow A_G^*(X)_{\mathbb{Q}}$  is a homomorphism of  $\psi$  (and therefore  $\lambda$ ) rings.

In order to define inertial Chern classes and the inertial  $\lambda$ - and  $\psi$ -ring structures, we will need the so-called *Bott cannibalistic classes*.

**Definition 3.15.** Let  $Y$  be an algebraic space with a proper action of a linear algebraic group  $G$ . Denote by  $K_G^+(Y)$  the semigroup of classes of  $G$ -equivariant vector bundles on  $Y$ .

For each  $j \geq 1$ , the  $j$ -th *Bott (cannibalistic) class*  $\theta^j : K_G^+(Y) \longrightarrow K_G(Y)$  is the multiplicative class defined for any line bundle  $\mathcal{L}$  by

$$\theta^j(\mathcal{L}) = \frac{1 - \mathcal{L}^j}{1 - \mathcal{L}} = \sum_{i=0}^{j-1} \mathcal{L}^i. \quad (19)$$

By the splitting principle, we can extend the definition of  $\theta^j(\mathcal{F})$  to all  $\mathcal{F}$  in  $K_G^+(Y)$ .

We will need to define Bott classes on elements of integral rank in rational K-theory. This can be done in a straightforward manner using a binomial expansion, but the resulting Bott class will lie in the augmentation completion of rational K-theory.

**Definition 3.16.** Let  $\mathfrak{a}_Y$  denote the kernel of the augmentation  $\epsilon : K_G(Y) \longrightarrow K_G(Y)$ . It is an ideal in the ring  $(K_G(Y), \cdot)$ , where  $\cdot$  denotes the usual tensor product, and it defines a topology on  $K_G(Y)$ . We denote the completion with respect to that topology by  $\widehat{K}_G(Y)_{\mathbb{Q}}$ .

The following proposition is straightforward.

**Proposition 3.17.** *For elements of  $K_G(Y)_{\mathbb{Q}}$  of integral rank, we can use a binomial expansion to define the Bott class as an element of the augmentation completion  $\widehat{K}_G(Y)_{\mathbb{Q}}$ .*

We will also need the following result.

**Theorem 3.18** (The Adams-Riemann-Roch Theorem for Equivariant Regular Embeddings [Köc91, Köc98]). *Let  $\iota : Y \hookrightarrow X$  be a  $G$ -equivariant closed regular embedding of smooth manifolds, then the following diagram commutes for all integers  $n \geq 1$ :*

$$\begin{array}{ccc} K_G(Y) & \xrightarrow{\theta^n(N_\iota^*)\psi^n} & K_G(Y) \\ \downarrow \iota_* & & \downarrow \iota_* \\ K_G(X) & \xrightarrow{\psi^n} & K_G(X), \end{array} \quad (20)$$

where  $N_\iota^*$  is the conormal bundle of the embedding  $\iota$ .

#### 4. AUGMENTATION IDEALS AND COMPLETIONS OF INERTIAL $K$ -THEORY

We will use the Bott classes of  $\mathcal{S}$  to define inertial  $\lambda$ - and  $\psi$ -ring structures as well as inertial Chern classes. Since  $\mathcal{S}$  is generally not integral, we will often need to work in the augmentation completion  $\widehat{K}_G(I_G X)_\mathbb{Q}$ . However, it is not *a priori* clear that the inertial product behaves well with respect to this completion, since the topology involved is constructed by taking classical powers of the classical augmentation ideal instead of inertial powers of the inertial augmentation ideal. The surprising result of this section is that these two completions are the same.

**Definition 4.1.** Given any inertial pair  $(\mathcal{R}, \mathcal{S})$ , define  $\mathfrak{a}_\mathcal{S}$  to be the kernel of the inertial augmentation  $\tilde{\epsilon} : K_G(I_G X) \longrightarrow K_G(I_G X)$ . It is an ideal with respect to the inertial product  $\star := \star_\mathcal{R}$ . Define  $\mathfrak{a}_{I_G X}$  to be the kernel of the classical augmentation  $\epsilon : K_G(I_G X) \longrightarrow K_G(I_G X)$ . It is an ideal of  $K_G(I_G X)$  with respect to the usual tensor product instead of the inertial product.

Each of these two ideals induces a topology on  $K_G(I_G X)_\mathbb{Q}$ , and we also consider a third topology induced by the augmentation ideal  $\mathfrak{a}_{\mathcal{B}G}$  of  $R(G)$ . By [EG00, Theorem 6.1a] the  $\mathfrak{a}_{\mathcal{B}G}$ -adic and  $\mathfrak{a}_{I_G X}$ -adic topologies on  $K_G(I_G X)_\mathbb{Q}$  are the same. In this section we will show that the  $\mathfrak{a}_\mathcal{S}$ -adic topology agrees with the other two.

**Lemma 4.2.** *If  $(\mathcal{R}, \mathcal{S})$  is an inertial pair, then  $(K_G(I_G X), \star_\mathcal{R})$  is an  $R(G)$ -algebra.*

*Proof.* By definition of an inertial pair, if  $\alpha_1 \in K_G(X)$  is supported in the untwisted sector and  $\beta_\Psi \in K_G(I(\Psi))$ , then  $\alpha_1 \star_\mathcal{R} \beta_\Psi = f_\Psi^* \alpha \cdot \beta$ , where  $f_\Psi : I(\Psi) \longrightarrow X$  is the projection.

The projection formula for equivariant  $K$ -theory implies that if  $x \in R(G)$ , then  $x\beta_\Psi = x\mathbf{1} \star_\mathcal{R} \beta_\Psi$ . Hence  $K_G(I_G X, \star_\mathcal{R})$  is an  $R(G)$ -algebra.  $\square$

**Theorem 4.3.** *The  $\mathfrak{a}_{\mathcal{B}G}$ -adic,  $\mathfrak{a}_{I_G X}$ -adic, and  $\mathfrak{a}_\mathcal{S}$ -adic topologies on  $K_G(I_G X)_\mathbb{Q}$  are all equivalent. In particular the  $\mathfrak{a}_{\mathcal{B}G}$ -adic, the  $\mathfrak{a}_{I_G X}$ -adic, and the  $\mathfrak{a}_\mathcal{S}$ -adic completions of  $K_G(I_G X)_\mathbb{Q}$  are equal.*

*Proof.* The main application of Theorem 4.3 is when  $G$  is diagonalizable, so we will give the proof only in that case. The case for general  $G$  is similar but notationally more complicated. To prove that the topologies are equivalent we must show that two conditions are satisfied.

- (1) For each positive integer  $n$  there is a positive integer  $r$ , such that  $\mathfrak{a}_{\mathcal{B}G}^{\otimes r} K_G(I_G X)_{\mathbb{Q}} \subseteq (\mathfrak{a}_{\mathcal{S}})^{\star n}$ .
- (2) For each positive integer  $n$  there is a positive integer  $r$ , such that  $(\mathfrak{a}_{\mathcal{S}})^{\star r} \subseteq \mathfrak{a}_{\mathcal{B}G}^{\otimes n} K_G(I_G X)_{\mathbb{Q}}$ .

Condition (1) follows from the observation that  $\mathfrak{a}_{\mathcal{B}G} K_G(I_G X) \subset \mathfrak{a}_{\mathcal{S}}$  and the fact that  $(K(I_G X), \star)$  is an  $R(G)$ -algebra. In particular we may take  $r = n$ .

Condition (2) is more difficult to check. Given a  $G$ -space  $Y$ , we denote by  $\mathfrak{a}_Y$  the subgroup of  $K_G(Y)$  of elements of rank 0. This is an ideal with respect to the tensor product.

For each connected component  $[U/G]$  of  $[I_G X/G]$ , the inertial augmentation satisfies  $\widetilde{\mathcal{H}}_0(\alpha)|_U = 0$  if  $\text{age}_{\mathcal{S}}([U/G]) > 0$  and  $\widetilde{\mathcal{H}}_0(\alpha)|_U = \text{Ch}_0(\alpha)|_U$  if  $\text{age}_{\mathcal{S}}([U/G]) = 0$  [EJK12, Thm 2.3.9]. So the ideal  $\mathfrak{a}_{\mathcal{S}}$  has the following decomposition as an Abelian group

$$\mathfrak{a}_{\mathcal{S}} = \bigoplus_{\text{age}_{\mathcal{S}}([U/G])=0} \mathfrak{a}_U \oplus \bigoplus_{\text{age}_{\mathcal{S}}([U/G])>0} K_G(U).$$

**Lemma 4.4.** *If  $m \in G$  with  $\alpha \in K_G(X^m) \cap \mathfrak{a}_{\mathcal{S}}$ , and  $\beta \in K_G(X^{m^{-1}}) \cap \mathfrak{a}_{\mathcal{S}}$ , then  $\alpha \star \beta \in \mathfrak{a}_{I_G X}$ .*

*Proof.* Since  $mm^{-1} = 1$ , we have  $\alpha \star \beta \in K_G(X^1) \subset K_G(I_G X)$ , so we must show that  $\alpha \star \beta \in \mathfrak{a}_X$ . First note that if  $\text{age}_{\mathcal{S}}[X^m/G] = 0$ , then  $\alpha_m \in \mathfrak{a}_{X^m}$ , so the inertial product

$$\mu_*(e_1^* \alpha \cdot e_2^* \beta \cdot \text{eu}(\mathcal{R}))$$

would automatically be in  $\mathfrak{a}_X$  because the finite pushforward  $\mu_*$  preserves the classical augmentation ideal.

Thus we may assume that  $\text{age}_{\mathcal{S}}([X^m/G])$  and  $\text{age}_{\mathcal{S}}([X^{m^{-1}}/G])$  are both non-zero and that  $\alpha$  and  $\beta$  have non-zero rank as elements of  $K_G(X^m)$  and  $K_G(X^{m^{-1}})$ , respectively. If the fixed locus  $X^{m, m^{-1}}$  has positive codimension, then  $\mu_*: K_G(X^{m, m^{-1}}) \subset K_G(X)$  is also in the classical augmentation ideal, since it consists of classes supported on subspaces of positive codimension. On the other hand, if  $X^{m, m^{-1}} = X$ , then  $T\mu|_{X^{m, m^{-1}}} = 0$ . By definition of an inertial pair,  $\mathcal{S}|_{X^1} = 0$ , so  $\mathcal{R}|_{X^{m, m^{-1}}} = e_1^* \mathcal{S} + e_2^* \mathcal{S}$  is a non-zero vector bundle. It follows that  $\text{eu}(\mathcal{R}|_{X^{m, m^{-1}}}) \in \mathfrak{a}_{X^{m, m^{-1}}}$ , and once again  $\alpha \star \beta \in \mathfrak{a}_X$ .  $\square$

Since  $G$  is diagonalizable and acts with finite stabilizer on  $X$ , there is a finite Abelian subgroup  $H \subset G$  such that  $X^g = \emptyset$  for all  $g \notin H$ . Let  $N$  be the maximum order of an element of  $H$ , and let  $s = |H|^N$ .

**Lemma 4.5.** *The  $(s+1)$ -fold inertial product  $(\mathfrak{a}_{\mathcal{S}})^{\star(s+1)}$  is contained in  $\mathfrak{a}_{I_G X}$ .*

*Proof.* By definition of the integer  $s$ , any  $s+1$ -fold product  $m_1 m_2 \cdots m_{s+1}$  in  $H$  can be written as  $mm^{-1}m'$  for some  $m$ , and  $m'$  in  $H$ . Correspondingly, we may write any product of the form  $\alpha_{m_1} \star \cdots \star \alpha_{m_{s+1}}$  with  $\alpha_{m_i} \in K_G(X^{m_i})$  as  $\alpha_m \star \beta_{m^{-1}} \star \gamma_{m'}$  for some  $\alpha_m \in K_G(X^m)$ ,  $\beta_m \in K_G(X^{m^{-1}})$ , and  $\gamma_{m'} \in K_G(X^{m'})$ . The lemma now follows from Lemma 4.4  $\square$

To complete the proof, observe first that we may use the equivalence of the  $\mathfrak{a}_{\mathcal{B}G}$ -adic and the  $\mathfrak{a}_{\mathbb{I}_{\mathcal{S}}^2}$ -adic topologies in the ring  $(K_G(\mathbb{I}_G^2 X), \otimes)$  to see that for any

$n$  there is an  $r$  such that  $\mathfrak{a}_{\mathbb{I}_G^2 X}^{\otimes r} \subset \mathfrak{a}_{\mathcal{B}G}^{\otimes n} K_G(\mathbb{I}_G^2 X)$ . This implies that  $\mu_*(\mathfrak{a}_{\mathbb{I}_G^2 X}^{\otimes r}) \subset \mathfrak{a}_{\mathcal{B}G}^{\otimes n} K_G(I_G X)$ . It follows that  $\mathfrak{a}_{\mathcal{S}}^{*(r(s+1))} \subset \mathfrak{a}_{\mathcal{B}G}^{\otimes n} K_G(I_G X)$ , as required.  $\square$

Since the three completions are all equal, we will not distinguish between them from now on, but will call them all the *augmentation completion* and denote them by  $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ . Note that this completion is a summand in  $K_G(I_G X)_{\mathbb{Q}}$ .

## 5. INERTIAL CHERN CLASSES AND POWER OPERATIONS

In this section we show that for each Gorenstein inertial pair  $(\mathcal{R}, \mathcal{S})$  and corresponding Chern Character  $\widetilde{\mathcal{Ch}}$ , we can define inertial Chern classes. When  $(\mathcal{R}, \mathcal{S})$  is strongly Gorenstein, there are also  $\psi$ -,  $\lambda$ -, and  $\gamma$ -operations on the corresponding inertial K-theory ring  $K_G(I_G X)$ . These operations behave nicely with respect to the inertial Chern character and satisfy many relations, including an analog of Theorem 3.13. In addition, when  $G$  is diagonalizable these operations make the inertial K-theory ring  $K_G(I_G X)$  into a  $\psi$ -ring and (after tensoring with  $\mathbb{Q}$ ) a  $\lambda$ -ring.

**5.1. Inertial Adams (power) operations and inertial Chern classes.** We begin by defining inertial Chern classes. We then define inertial Adams operations associated to a strongly Gorenstein inertial pair  $(\mathcal{R}, \mathcal{S})$  and show that, for a diagonalizable group  $G$ , the corresponding rings are  $\psi$ -rings and have many other nice properties.

**Definition 5.1.** For any Gorenstein inertial pair  $(\mathcal{R}, \mathcal{S})$  the  $\mathcal{S}$ -inertial Chern series

$$\widetilde{c}_t : K_G(I_G X) \longrightarrow A_G^*(I_G X)_{\mathbb{Q}}[[t]]$$

is defined, for all  $\mathcal{F}$  in  $K_G(I_G X)$ , by

$$\widetilde{c}_t(\mathcal{F}) = \widetilde{\exp} \left( \sum_{n \geq 1} (-1)^{n-1} (n-1)! t^n \widetilde{\mathcal{Ch}}^n(\mathcal{F}) \right), \quad (21)$$

where the exponential power series  $\widetilde{\exp}$  is defined with respect to the  $\star_{\mathcal{R}}$  product and  $\widetilde{\mathcal{Ch}}^n(\mathcal{F})$  is the component of  $\widetilde{\mathcal{Ch}}(\mathcal{F})$  in  $A^*(I_G X)$  with  $\mathcal{S}$ -age equal to  $n$ . For all  $i \geq 0$ , the  $i$ -th  $\mathcal{S}$ -inertial Chern class  $\widetilde{c}^i(\mathcal{F})$  of  $\mathcal{F}$  is the coefficient of  $t^i$  in  $\widetilde{c}_t(\mathcal{F})$ .

**Remark 5.2.** The definition of inertial Chern classes could be extended to the non-Gorenstein case by introducing fractionally graded  $\mathcal{S}$ -inertial Chern classes, but the latter does not behave nicely with respect to the inertial  $\psi$ -structures.

**Definition 5.3.** Let  $(\mathcal{R}, \mathcal{S})$  be a strongly Gorenstein inertial pair. For all integers  $j \geq 1$ , we define the  $j$ -th inertial Adams (or power) operation  $\widetilde{\psi}^j : K_G(I_G X) \longrightarrow K_G(I_G X)$  by the formula

$$\widetilde{\psi}^j(\mathcal{F}) := \psi^j(\mathcal{F}) \cdot \theta^j(\mathcal{S}^*) \quad (22)$$

for all  $\mathcal{F}$  in  $K_G(I_G X)$ . (Here  $\cdot$  is the ordinary product on  $K_G(I_G X)$ .)

**Remark 5.4.** If  $(\mathcal{R}, \mathcal{S})$  is Gorenstein, then  $\mathcal{S}$  has integral rank, and Proposition 3.17 states that  $\theta^j(\mathcal{S}^*)$  may be defined as an element of the completion  $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ , so we can still define inertial Adams operations as maps  $\widetilde{\psi}^j : K_G(I_G X) \longrightarrow \widehat{K}_G(I_G X)_{\mathbb{Q}}$ .

**Definition 5.5.** Let  $(\mathcal{R}, \mathcal{S})$  be a strongly Gorenstein inertial pair. We define  $\tilde{\lambda}_t : K_G(I_G X) \longrightarrow K_G(I_G X)_{\mathbb{Q}}[[t]]$  by Equation (10) after replacing  $\psi$ ,  $\lambda$ , and  $\exp$  by their respective inertial analogs  $\tilde{\psi}$ ,  $\tilde{\lambda}$ , and  $\widetilde{\exp}$ :

$$\tilde{\lambda}_t = \widetilde{\exp} \left( \sum_{r \geq 1} (-1)^{r-1} \tilde{\psi}^r \frac{t^r}{r} \right), \quad (23)$$

Further, define  $\tilde{\lambda}^i$  to be the coefficient of  $t^i$  in  $\tilde{\lambda}_t$ . We call  $\tilde{\lambda}^i$  the  $i$ -th inertial  $\lambda$  operation.

We will now prove a relation between the inertial Chern classes, inertial Chern character, and the inertial Adams operations, but to do this, we first need two lemmas connecting the classical Chern character, Adams operations, Bott classes, and Todd classes.

**Lemma 5.6.** Let  $\mathcal{F} \in K_G(I_G X)$  be the class of a  $G$ -equivariant vector bundle on  $I_G X$ . For all integers  $n \geq 1$ , we have the equality in  $A_G^*(I_G X)$ :

$$\text{Ch}(\theta^n(\mathcal{F}^*)) \text{Td}(-\mathcal{F}) = n^{\text{Ch}^0(\mathcal{F})} \text{Td}(-\psi^n(\mathcal{F})). \quad (24)$$

More generally, if  $\mathcal{F} \in K_G(I_G X)_{\mathbb{Q}}$  is such that  $\mathcal{F} = \sum_{i=1}^k \alpha_i \mathcal{V}_i$ , where  $\mathcal{V}_i$  is a vector bundle,  $\alpha_i \in \mathbb{Q}$  with  $\alpha_i > 0$  for all  $i = 1, \dots, k$ , and  $\text{Ch}^0(\mathcal{F}) \in \mathbb{Z}^{\ell} \subset A_G^0(I_G X)_{\mathbb{Q}}$ , ( $\ell$  is the number of connected components of  $[I_G X/G]$ ), then Equation (24) still holds in  $A_G^*(I_G X)_{\mathbb{Q}}$ , where  $\theta^n(\mathcal{F}^*)$  is interpreted as an element in the completion  $\hat{K}_G(I_G X)_{\mathbb{Q}}$ .

*Proof.* Let  $\mathcal{L}$  in  $K_G(I_G X)$  be a line bundle with ordinary first Chern class  $c := c^1(\mathcal{L})$ . For all  $n \geq 1$  we have

$$\begin{aligned} \text{Ch}(\theta^n(\mathcal{L}^*)) \text{Td}(-\mathcal{L}) &= \text{Ch} \left( \frac{1 - (\mathcal{L}^*)^n}{1 - \mathcal{L}^*} \right) (\text{Td}(\mathcal{L}))^{-1} \\ &= \left( \frac{1 - e^{-nc}}{1 - e^{-c}} \right) \left( \frac{c}{1 - e^{-c}} \right)^{-1} \\ &= n \left( \frac{nc}{1 - e^{-nc}} \right)^{-1} \\ &= n \text{Td}(\mathcal{L}^n)^{-1}, \end{aligned}$$

and we conclude that

$$\text{Ch}(\theta^n(\mathcal{L}^*)) \text{Td}(-\mathcal{L}) = n \text{Td}(-\psi^n(\mathcal{L})).$$

Equation (24) now follows from the splitting principle, the multiplicativity of  $\theta^n$  and  $\text{Td}$ , and the fact that  $\text{Ch}$  is a ring homomorphism.

The more general statement follows the fact that  $\text{Ch}$  and  $\text{Td}$  factor through  $\hat{K}_G(I_G X)_{\mathbb{Q}}$ , together with the fact that  $\text{Ch}^0(\theta^j(\mathcal{F}) - j^{\epsilon(\mathcal{F})}) = 0$ .  $\square$

We now define a K-theoretic Todd class that will be important for the proof of Theorem 5.16.

**Definition 5.7.** Let  $\mathrm{Td}^K : K_G(I_G X) \longrightarrow \widehat{K}_G(I_G X)_{\mathbb{Q}}$  be the multiplicative class such that if  $\mathcal{L}$  is an equivariant line bundle, then

$$\mathrm{Td}^K(\mathcal{L}) = \frac{\mathcal{L} \log \mathcal{L}}{\mathcal{L} - 1} = 1 + \frac{1}{2}(\mathcal{L} - 1) - \frac{1}{6}(\mathcal{L} - 1)^2 + \cdots. \quad (25)$$

The map  $\mathrm{Td}^K$  factors through the obvious projection  $K_G(I_G X) \longrightarrow \widehat{K}_G(I_G X)_{\mathbb{Q}}$ , so we use the same notation to denote the corresponding map  $\widehat{K}_G(I_G X)_{\mathbb{Q}} \longrightarrow \widehat{K}_G(I_G X)_{\mathbb{Q}}$ .

A straightforward calculation gives the next lemma.

**Lemma 5.8.** *For all  $\mathcal{F}$  in  $K_G(I_G X)_{\mathbb{Q}}$ , we have the identity*

$$\mathrm{Ch}(\mathrm{Td}^K(\mathcal{F})) = \mathrm{Td}(\mathcal{F}). \quad (26)$$

**Theorem 5.9.** *Let  $(\mathcal{R}, \mathcal{S})$  be a strongly Gorenstein inertial pair. Then for any  $\alpha \in \mathbb{N}$  and integer  $n \geq 1$*

$$\widetilde{\mathcal{Ch}}^{\alpha}(\widetilde{\psi}^n(\mathcal{F})) = n^{\alpha} \widetilde{\mathcal{Ch}}^{\alpha}(\mathcal{F}) \quad (27)$$

in  $A_G^{\{\alpha\}}(I_G X)$ , where the grading is the  $\mathcal{S}$ -age grading.

**Remark 5.10.** If  $(\mathcal{R}, \mathcal{S})$  is on a Gorenstein inertial pair, then Equation (27) holds in  $A_G^{\{\alpha\}}(I_G X)_{\mathbb{Q}}$ , where  $\widetilde{\psi}^n$  is interpreted as a map  $\widetilde{\psi}^n : K_G(I_G X) \longrightarrow \widehat{K}_G(I_G X)_{\mathbb{Q}}$  (cf. Remark 5.4).

*Proof.*

$$\begin{aligned} \widetilde{\mathcal{Ch}}(\widetilde{\psi}^n(\mathcal{F})) &= \mathrm{Ch}(\psi^n(\mathcal{F})\theta^n(\mathcal{S}^*)) \mathrm{Td}(-\mathcal{S}) \\ &= \mathrm{Ch}(\psi^n(\mathcal{F}))\mathrm{Ch}(\theta^n(\mathcal{S}^*)) \mathrm{Td}(-\mathcal{S}) \\ &= \mathrm{Ch}(\psi^n(\mathcal{F})) \mathrm{Td}(-\psi^n(\mathcal{S})) n^{\mathrm{age}} \\ &= n^{\mathrm{age}} \mathrm{Ch}(\psi^n(\mathcal{F}) \mathrm{Td}^K(-\psi^n(\mathcal{S}))) \\ &= n^{\mathrm{age}} \mathrm{Ch}(\psi^n(\mathcal{F}) \psi^n(\mathrm{Td}^K(-\mathcal{S}))) \\ &= n^{\mathrm{age}} \mathrm{Ch}(\psi^n(\mathcal{F}) \mathrm{Td}^K(-\mathcal{S})) \\ &= n^{\mathrm{age}} \sum_{i \geq 0} \mathrm{Ch}^i(\psi^n(\mathcal{F}) \mathrm{Td}^K(-\mathcal{S})) \\ &= \sum_{i \geq 0} n^{\mathrm{age}} n^i \mathrm{Ch}^i(\mathcal{F} \mathrm{Td}^K(-\mathcal{S})) \\ &= \sum_{\alpha \in \mathbb{N}} n^{\alpha} \widetilde{\mathcal{Ch}}^{\alpha}(\mathcal{F}), \end{aligned}$$

where the third equality follows from Equation (24), the fourth from the definition of  $\mathrm{Td}^K$  and then fifth from the fact that  $\psi^n$  and  $\mathrm{Td}^K$  commute modulo the kernel of the completion map  $K_G(I_G X) \longrightarrow \widehat{K}_G(I_G X)$ ; the sixth follows since  $\psi^n$  preserves the ordinary product, the seventh from Equation (16). The final equality follows from Equation (26), the fact that  $\mathrm{Ch}$  preserves the ordinary product, and the definition of  $\widetilde{\mathcal{Ch}}^{\alpha}$ .

The second part of the theorem follows from the fact that  $\widetilde{\mathcal{Ch}}$  factors through the completion  $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ .  $\square$

**Definition 5.11.** Let  $(\mathcal{R}, \mathcal{S})$  be a strongly Gorenstein inertial pair. We define the inertial operations  $\tilde{\gamma}_t$  on inertial K-theory as in Equation (15), that is

$$\tilde{\gamma}_t := \sum_{i=0}^{\infty} t^i \tilde{\gamma}^i := \tilde{\lambda}_{t/(1-t)}. \quad (28)$$

**Remark 5.12.** If  $(\mathcal{R}, \mathcal{S})$  is only Gorenstein, then we may still define  $\gamma_t$  as a map  $K_G(I_G X) \longrightarrow \hat{K}_G(I_G X)[[t]]$ .

**Theorem 5.13.** Let  $(\mathcal{R}, \mathcal{S})$  be a Gorenstein inertial pair. The  $\mathcal{S}$ -inertial Chern series  $\tilde{c}_t : K_G(I_G X) \longrightarrow A_G^*(I_G X)_{\mathbb{Q}}$  satisfies the following properties:

**Consistency with  $\tilde{\gamma}$ :** For all integers  $n \geq 1$  and for all  $\mathcal{F}$  in  $K_G(I_G X)_{\mathbb{Q}}$ , we have the following equality in  $A_G^*(I_G X)_{\mathbb{Q}}$

$$\tilde{c}^n(\mathcal{F}) = \tilde{\mathcal{H}}^n(\tilde{\gamma}^n(\mathcal{F} - \tilde{\epsilon}(\mathcal{F}))), \quad (29)$$

where  $\tilde{\gamma}_t$  is interpreted as a map  $K_G(I_G X)_{\mathbb{Q}} \longrightarrow \hat{K}_G(I_G X)_{\mathbb{Q}}[[t]]$ .

**Multiplicativity:** For all  $\mathcal{V}$  and  $\mathcal{W}$  in  $K_G(I_G X)_{\mathbb{Q}}$ ,

$$\tilde{c}_t(\mathcal{V} + \mathcal{W}) = \tilde{c}_t(\mathcal{V}) \star_{\mathcal{R}} \tilde{c}_t(\mathcal{W}).$$

**Zeroth Chern class:** For all  $\mathcal{V}$  in  $K_G(I_G X)_{\mathbb{Q}}$ ,

$$\tilde{c}^0(\mathcal{V}) = \mathbf{1}.$$

**Untwisted sector:** For all  $\mathcal{F}$  supported on the untwisted sector (i.e.,  $\mathcal{F} \in K_G(X^1) \subseteq K_G(I_G X)$ ), the inertial Chern classes agree with the ordinary Chern classes, i.e.,

$$\tilde{c}_t(\mathcal{F}) = c_t(\mathcal{F}). \quad (30)$$

**Classes of Unity:** All the inertial Chern classes of unity vanish, except for  $\tilde{c}^0(\mathbf{1})$ :

$$\tilde{c}_t(\mathbf{1}) = \mathbf{1}. \quad (31)$$

**Remark 5.14.** The theorem shows that Equation (29) yields an alternative, but equivalent, definition of inertial Chern classes.

**Remark 5.15.** The previous theorem holds for a general inertial pair of a  $G$ -space  $X$ , provided that  $K_G(I_G X)$  and  $A_G^*(I_G X)$  are replaced by their Gorenstein subrings  $\hat{K}_G(I_G X)$  and  $\hat{A}_G^*(I_G X)$ , respectively.

*Proof.* Multiplicativity and  $\tilde{c}^0(\mathcal{V}) = \mathbf{1}$  follows immediately from the exponential form of Equation (21) and the fact that  $\tilde{\mathcal{H}}$  is a homomorphism.

On the untwisted sector, inertial products reduce to the ordinary products, and the inertial Chern character reduces to the classical Chern character, and this shows that Equation (21) agrees with Equation (17), which implies Equation (30).

Equation (31) will follow immediately from Equation (29).

The hard part of this proof is the consistency of the inertial Chern classes with  $\tilde{\gamma}$  (Equation (29)). To prove this, it will be useful to first introduce the ring homomorphism  $\tilde{\mathcal{H}}_t : K_G(I_G X) \longrightarrow A_G^*(I_G X)[[t]] \otimes \mathbb{Q}$  via

$$\tilde{\mathcal{H}}_t(\mathcal{F}) := \sum_{n \geq 0} \tilde{\mathcal{H}}^n(\mathcal{F}) t^n.$$

For the remainder of the proof, all products are understood to be inertial products. We have the following equality in  $\widetilde{K}_G(I_G X)[[t]]$ ,

$$\begin{aligned}
\widetilde{\mathcal{H}}_t(\widetilde{\lambda}_u(\mathcal{F})) &= \widetilde{\exp} \left( \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \widetilde{\mathcal{H}}_t(\widetilde{\psi}^k(\mathcal{F})) u^k \right) \\
&= \widetilde{\exp} \left( \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \widetilde{\mathcal{H}}_{kt}(\mathcal{F}) u^k \right) \\
&= \widetilde{\exp} \left( \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\alpha \geq 0} \widetilde{\mathcal{H}}^\alpha(\mathcal{F}) (kt)^\alpha u^k \right) \\
&= \widetilde{\exp} \left( \sum_{\alpha \geq 0} \widetilde{\mathcal{H}}^\alpha(\mathcal{F}) t^\alpha \sum_{k \geq 1} (-1)^{k-1} k^{\alpha-1} u^k \right),
\end{aligned}$$

where the first equality follows from the definition of  $\widetilde{\lambda}$  and the fact that  $\widetilde{\mathcal{H}}_u$  is a ring homomorphism, and the second equality follows from Equation (27). From the definition of  $\widetilde{\gamma}_t$ , it follows that

$$\begin{aligned}
\widetilde{\mathcal{H}}_t(\widetilde{\gamma}_u(\mathcal{F} - \widetilde{\epsilon}(\mathcal{F}))) &= \widetilde{\exp} \left( \sum_{\alpha \geq 0} \widetilde{\mathcal{H}}^\alpha(\mathcal{F} - \widetilde{\epsilon}(\mathcal{F})) t^\alpha \sum_{k \geq 1} (-1)^{k-1} k^{\alpha-1} \left( \frac{u}{1-u} \right)^k \right) \\
&= \widetilde{\exp} \left( \sum_{\alpha \geq 0} \sum_{k \geq 1} (-1)^{k-1} k^{\alpha-1} \widetilde{\mathcal{H}}^\alpha(\mathcal{F}) t^\alpha \sum_{n \geq k} u^n \binom{n-1}{k-1} \right) \\
&= \widetilde{\exp} \left( \sum_{\alpha \geq 0} \widetilde{\mathcal{H}}^\alpha(\mathcal{F}) t^\alpha \sum_{n \geq 1} u^n \sum_{k=1}^n (-1)^{k-1} k^{\alpha-1} \binom{n-1}{k-1} \right) \\
&= \widetilde{\exp} \left( \sum_{\alpha \geq 0} \widetilde{\mathcal{H}}^\alpha(\mathcal{F}) t^\alpha \sum_{n \geq 1} u^n (-1)^{n-1} (n-1)! S(\alpha, n) \right),
\end{aligned}$$

where

$$S(\alpha, n) = \frac{1}{n!} \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} j^\alpha$$

are the Stirling numbers of the second kind. Projecting out those terms which are not powers of  $z := ut$  yields the equality

$$\sum_{\ell \geq 0} \widetilde{\mathcal{H}}^\ell(\widetilde{\gamma}^\ell(\mathcal{F} - \widetilde{\epsilon}(\mathcal{F}))) z^\ell = \widetilde{\exp} \left( \sum_{s \geq 0} z^s \widetilde{\mathcal{H}}^s(\mathcal{F}) (-1)^{n-1} (n-1)! S(n, n) \right).$$

The identity  $S(n, n) = 1$  and Equation (21) yields Equation (29).  $\square$

**5.2.  $\psi$ - and  $\lambda$ -ring structures on inertial K-theory.** The main result of this section is the following:

**Theorem 5.16.** *Let  $G$  be a diagonalizable group and let  $(\mathcal{R}, \mathcal{S})$  be a strongly Gorenstein inertial pair on  $I_G X$ . The ring  $(K_G(I_G X), \star_{\mathcal{R}}, \mathbf{1}, \widetilde{\epsilon}, \widetilde{\psi})$  is an augmented  $\psi$ -ring.*

Moreover, for general (possibly non-diagonalizable)  $G$  and any inertial pair  $(\mathcal{R}, \mathcal{S})$ , the augmentation completion of the Gorenstein subring  $\check{K}_G(I_G X)$  of  $K_G(I_G X)$  is an augmented  $\psi$ -ring.

With a little work we get the following corollary.

**Corollary 5.17.** *Let  $(\mathcal{R}, \mathcal{S})$  be a strongly Gorenstein inertial pair with  $G$  diagonalizable. Then  $(K_G(I_G X)_{\mathbb{Q}}, \star_{\mathcal{R}}, \mathbf{1}, \tilde{\lambda})$  is an augmented  $\lambda$ -ring.*

Moreover, for general (possibly non-diagonalizable)  $G$  and any inertial pair  $(\mathcal{R}, \mathcal{S})$ , the augmentation completion of the Gorenstein subring  $\check{K}_G(I_G X)$  of  $K_G(I_G X)$  is an augmented  $\lambda$ -ring.

*Proof of Corollary 5.17.* Combining Theorem 5.16 with Theorem 3.3, all that we must prove is that

$$\tilde{\epsilon}(\tilde{\lambda}_t(\mathcal{F})) = \tilde{\lambda}_t(\tilde{\epsilon}(\mathcal{F})) = (1+t)^{\tilde{\epsilon}(\mathcal{F})}. \quad (32)$$

Here we have omitted the  $\star$  from the notation, but all products are the inertial product  $\star$ , and exponentiation is also with respect to the product  $\star$ .

For all  $\mathcal{F} \in K_G(I_G X)$ , we have

$$\tilde{\epsilon}(\tilde{\lambda}_t(\mathcal{F})) = \sum_{i \geq 0} t^i \tilde{\epsilon}(\tilde{\lambda}^i(\mathcal{F})),$$

but

$$\begin{aligned} \tilde{\epsilon}(\tilde{\lambda}_t(\mathcal{F})) &= \tilde{\epsilon}(\widetilde{\exp}\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} t^n \tilde{\psi}^n(\mathcal{F})\right)) \\ &= \widetilde{\exp}\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} t^n \tilde{\epsilon}(\tilde{\psi}^n(\mathcal{F}))\right) \\ &= \widetilde{\exp}\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} t^n \tilde{\epsilon}(\mathcal{F})\right) \\ &= (1+t)^{\tilde{\epsilon}(\mathcal{F})}, \end{aligned}$$

where we have used the fact that  $\tilde{\epsilon} \circ \tilde{\psi}^n = \tilde{\epsilon}$  (by Theorem 5.16) in the third line.

Finally, we also have that  $\tilde{\lambda}_t(\tilde{\epsilon}(\mathcal{F})) = (1+t)^{\tilde{\epsilon}(\mathcal{F})}$ , since  $\tilde{\epsilon}$  commutes with  $\tilde{\psi}$  by Theorem 5.16.  $\square$

**Remark 5.18.** The hypothesis that  $G$  is diagonalizable is necessary, as demonstrated by the following example.

Let  $G = S_3$  and let  $\mathcal{X} = \mathcal{B}S_3$ . There are two twisted sectors, corresponding to the two non-trivial conjugacy classes in  $S_3$ , and these sectors are isomorphic to  $\mathcal{B}\mu_2$  and  $\mathcal{B}\mu_3$ . The double inertia has five sectors, two isomorphic to  $\mathcal{B}\{e\}$  (corresponding to the conjugacy classes of the pairs  $((12), (123))$  and  $((123), (12))$ , respectively), one isomorphic to  $\mathcal{B}\mu_2$ , one isomorphic to  $\mathcal{B}\mu_3$ , and the untwisted sector isomorphic to  $\mathcal{B}S_3$ . Identifying  $K(\mathcal{B}G) = R(G)$ , the multiplication map  $\mu: \mathbb{I}^2 \mathcal{X} \longrightarrow I\mathcal{X}$  corresponds to the induced representation functor on each sector. Consider the inertial product with  $\mathcal{R} = 0$  and  $\mathcal{S} = 0$ . (This is just the usual orbifold product on  $\mathcal{B}S_3$ .) Let  $\chi \in R(\mu_2)$  be the defining character. Then

$$\begin{aligned} \chi|_{\mathcal{B}\mu_2} \star 1|_{\mathcal{B}\mu_2} &= (\text{Ind}_{\mu_2}^{S_3} \chi)|_{\mathcal{B}S_3} \\ &= (\text{sgn} + V_2)|_{\mathcal{B}S_3}, \end{aligned}$$

where  $\text{sgn}$  is the sign representation on  $S_3$ , and  $V_2$  is the 2-dimensional irreducible representation. Thus the character of  $\psi^2(\chi|_{\mathcal{B}\mu_2} \star 1|_{\mathcal{B}\mu_2})$  has value 3 at the identity and at the conjugacy class of a 2-cycle, and it has value 0 on 3-cycles. On the other hand,  $\psi^2(\chi) = \psi^2(1) = 1$  in  $R(\mu_2)$ , so  $\psi^2(\chi|_{\mathcal{B}\mu_2}) \star \psi^2(1|_{\mathcal{B}\mu_2}) = \text{Ind}_{\mu_2}^{S^3} 1 = 1 + V$ . The character of  $1 + V$  has value 1 on 2-cycles, so these classes are not isomorphic.

*Proof of Theorem 5.16.* First, it is straightforward from the definition that  $\tilde{\psi}^n(\mathcal{F} + \mathcal{G}) = \tilde{\psi}^n(\mathcal{F}) + \tilde{\psi}^n(\mathcal{G})$ , and also  $\tilde{\psi}^1(\mathcal{F}) = \mathcal{F}$ , since  $\theta^1(\mathcal{G}) = \mathbf{1}$  for any  $\mathcal{G}$ . Second, we have  $\tilde{\psi}^n(\mathbf{1}) = \mathbf{1}$ , since  $\mathbf{1}$  is supported only on  $K_G(X^1)$ , and  $\mathcal{S}_{X^1} = 0$  (because  $(\mathcal{R}, \mathcal{S})$  is an inertial pair). Now, to show for all  $\mathcal{F}$  in  $K_G(I_G X)$  that

$$\tilde{\psi}^n(\tilde{\psi}^\ell(\mathcal{F})) = \tilde{\psi}^{n\ell}(\mathcal{F}),$$

we observe that

$$\begin{aligned} \tilde{\psi}^n(\tilde{\psi}^\ell(\mathcal{F})) &= \tilde{\psi}^n(\psi^\ell(\mathcal{F})\theta^\ell(\mathcal{S}^*)) \\ &= \psi^{n\ell}(\mathcal{F})\psi^n(\theta^\ell(\mathcal{S}^*)\theta^n(\mathcal{S}^*)). \end{aligned}$$

Hence, we need to show that

$$\psi^n(\theta^\ell(\mathcal{S}^*))\theta^n(\mathcal{S}^*) = \theta^{n\ell}(\mathcal{S}^*).$$

This follows from the splitting principle, from the fact that the Bott classes are multiplicative, and from the fact that for any line bundle  $\mathcal{L}$  we have

$$\psi^n(\theta^\ell(\mathcal{L}))\theta^n(\mathcal{L}) = \psi^n\left(\frac{1 - \mathcal{L}^\ell}{1 - \mathcal{L}}\right) \frac{1 - \mathcal{L}^n}{1 - \mathcal{L}} = \frac{1 - \mathcal{L}^{n\ell}}{1 - \mathcal{L}^n} \frac{1 - \mathcal{L}^n}{1 - \mathcal{L}} = \theta^{n\ell}(\mathcal{L}). \quad (33)$$

It remains to show that  $\tilde{\psi}$  preserves the inertial product defined by  $\mathcal{R}$ , i.e.,

$$\tilde{\psi}^n(\mathcal{F} \star \mathcal{G}) = \tilde{\psi}^n(\mathcal{F}) \star \tilde{\psi}^n(\mathcal{G}), \quad (34)$$

where  $\star$  is understood to refer to the  $\star_{\mathcal{R}}$ -product. It is at this point in the proof that we need to use the hypothesis that  $G$  is diagonalizable.

**Lemma 5.19.** *If  $G$  is a diagonalizable group, then there is a covering of  $\mathbb{I}_G^2 X$  by open and closed subsets such that the restriction of the multiplication map  $\mu: \mathbb{I}_G^2 X \longrightarrow IX$  is a closed immersion.*

*Proof.* We know that there is a decomposition of  $\mathbb{I}_G^2 X$  into closed and open components indexed by conjugacy classes of pairs in  $G \times G$ . However, since  $G$  is diagonalizable, each conjugacy class consists of a single element. If  $\Psi = \{(m_1, m_2)\}$ , then  $\mathbb{I}^2(\Psi) = X^{g,h}$  and the multiplication map restricts to the closed embedding  $\mu: X^{g,h} \longrightarrow X^{gh}$ .  $\square$

Given the lemma, let us prove that  $\tilde{\psi}$  is compatible with the inertial product.

$$\begin{aligned} \tilde{\psi}^n(\mathcal{V} \star \mathcal{W}) &= \theta^n(\mathcal{S}^*) \cdot \psi^n(\mathcal{V} \star \mathcal{W}) \\ &= \theta^n(\mathcal{S}^*) \cdot \psi^n(\mu_*(e_1^* \mathcal{V} \cdot e_2^* \mathcal{W} \cdot \lambda_{-1}(\mathcal{R}^*))) \end{aligned} \quad (35)$$

By our Lemma,  $\mu_*$  is a closed embedding, so by the equivariant Adams-Riemann-Roch for closed embeddings (Theorem 3.18), we have  $\psi^n \mu_* \alpha = \mu_*(\theta^n(N_\mu^*) \psi^n \alpha)$ ,

where  $N_\mu^*$  is the conormal bundle of  $\mu$ . Writing  $N_\mu^* = -T_\mu^*$  we obtain the equalities

$$\begin{aligned}
&= \theta^n(\mathcal{S}^*) \cdot \mu_* [\theta^n(-T^*\mu) \cdot \psi^n(e_1^*\mathcal{V} \cdot e_2^*\mathcal{W} \cdot \lambda_{-1}(\mathcal{R})^*)] \\
&= \theta^n(\mathcal{S}^*) \cdot \mu_* [\theta^n(-T^*\mu) \cdot e_1^*\psi^n(\mathcal{V}) \cdot e_2^*\psi^n(\mathcal{W}) \cdot \psi^n(\lambda_{-1}(\mathcal{R}^*))] \\
&= \theta^n(\mathcal{S}^*) \cdot \mu_* [\theta^n(-T^*\mu) \cdot e_1^*\psi^n(\mathcal{V}) \cdot e_2^*\psi^n(\mathcal{W}) \cdot \lambda_{-1}(\psi^n(\mathcal{R}^*))] \\
&= \theta^n(\mathcal{S}^*) \cdot \mu_* [\theta^n(-T^*\mu) \cdot e_1^*\psi^n(\mathcal{V}) \cdot e_2^*I\pi^*\psi^n(\mathcal{W}) \cdot \lambda_{-1}(\mathcal{R}^*) \cdot \theta^n(\mathcal{R}^*)] \\
&= \theta^n(\mathcal{S}^*) \cdot \mu_* [e_1^*\psi^n(\mathcal{V}) \cdot e_2^*\psi^n(\mathcal{W}) \cdot \lambda_{-1}(\mathcal{R}^*) \cdot \theta^n(\mathcal{R}^* - T^*\mu)],
\end{aligned} \tag{36}$$

where the second equality follows from the fact that  $\psi^n$  respects the ordinary  $(\cdot)$  multiplication, the third from the definition of the Euler class and the fact that [Knu73, p. 48] for all  $i, n$ ,

$$\psi^n \circ \lambda^i = \lambda^i \circ \psi^n,$$

the fourth from the fact that for any non-negative element  $\mathcal{F}$  in  $\tilde{K}_G(I_G X)$  we have

$$\theta^n(\mathcal{F})\lambda_{-1}(\mathcal{F}) = \lambda_{-1}(\psi^n(\mathcal{F})),$$

and the fifth from the multiplicativity of  $\theta^n$ . Since  $\tilde{\psi}^n(\mathcal{F}) = \psi^n(\mathcal{F})\theta^n(\mathcal{S}^*)$ , we may express the last line of (36) as

$$\theta^n(\mathcal{S}^*)\mu_* \left[ e_1^*\tilde{\psi}^n(\mathcal{V}) \cdot e_2^*\tilde{\psi}^n(\mathcal{W}) \cdot \lambda_{-1}(\mathcal{R}^*) \cdot \theta^n(\mathcal{R}^* - T^*\mu - e_1^*\mathcal{S}^* - e_2^*\mathcal{S}^*) \right]. \tag{37}$$

Applying the projection formula to (37) yields

$$\tilde{\psi}^n(\mathcal{V} \star \mathcal{W}) = \mu_* \left[ e_1^*\tilde{\psi}^n(\mathcal{V}) \cdot e_2^*\tilde{\psi}^n(\mathcal{W}) \cdot \lambda_{-1}(\mathcal{R}^*) \cdot \theta^n(\mathcal{R}^* - T^*\mu - e_1^*\mathcal{S}^* - e_2^*\mathcal{S}^* + \mu^*\mathcal{S}^*) \right]. \tag{38}$$

Now because  $(\mathcal{R}, \mathcal{S})$  is an inertial pair, we have

$$\mathcal{R} = e_1^*\mathcal{S} + e_2^*\mathcal{S} - \mu^*\mathcal{S} + T\mu,$$

so

$$\begin{aligned}
\tilde{\psi}^n(\mathcal{V} \star \mathcal{W}) &= \mu_* \left[ e_1^*\tilde{\psi}^n(\mathcal{V}) \cdot e_2^*\tilde{\psi}^n(\mathcal{W}) \cdot \lambda_{-1}(\mathcal{R}^*) \right] \\
&= \tilde{\psi}(\mathcal{V}) \star \tilde{\psi}(\mathcal{W}),
\end{aligned}$$

as claimed.

Finally, the definition of  $\tilde{\psi}$  and the fact that the ordinary augmentation in ordinary equivariant K-theory is preserved by and commutes with the ordinary  $\psi$  operations, we have

$$\tilde{\epsilon}(\tilde{\psi}^n(\mathcal{V})) = \tilde{\psi}^n(\tilde{\epsilon}(\mathcal{V})) = \tilde{\epsilon}(\mathcal{V}) \tag{39}$$

In the case where  $G$  is not diagonalizable,  $\mu_*$  is not an embedding, but the equivariant Adams-Riemann-Roch theorem nevertheless holds after completing at the augmentation ideal. Restricting to the augmentation completion of the Gorenstein subring insures that the Bott class  $\theta^k(\mathcal{S}^*)$  takes values in that subring (which has  $\mathbb{Q}$  coefficients), whereas the Bott class in general would take values in the augmentation completion of  $K_G(I_G X) \otimes \overline{\mathbb{Q}}$ . The rest of the above argument goes through verbatim.  $\square$

**Remark 5.20.** After extending scalars the  $\psi$ -ring structure on the augmentation completion  $\widehat{K}_G(I_G X)_{\mathbb{C}}$  can be identified with  $K_G(X)_{\mathbb{C}}$ , and so we obtain a new  $\psi$ -ring structure on ordinary equivariant K-theory. These operations will be investigated in a subsequent paper.

**Remark 5.21.** Let  $(\mathcal{R}, \mathcal{S})$  be a Gorenstein inertial pair on  $I_G X$ . For each integer  $k \geq 1$ , let  $\tilde{\psi}^k : A_G^*(I_G X) \longrightarrow A_G^*(I_G X)$  be defined by Equation (14) for all  $v$  in  $A_G^{\{d\}}(I_G X)$ . If  $\tilde{\epsilon} : A_G^*(I_G X) \longrightarrow A_G^{\{0\}}(I_G X)$  is the canonical projection, then the inertial Chow theory  $(A_G^*(I_G X), \star, 1, \tilde{\psi}, \tilde{\epsilon})$  is an augmented  $\psi$ -ring.

Moreover, if  $G$  is a diagonalizable group and  $(\mathcal{R}, \mathcal{S})$  is a strongly Gorenstein inertial pair on  $I_G X$ , then the summand  $\hat{K}_G(I_G X)_{\mathbb{Q}}$  inherits an augmented  $\psi$ -ring structure from  $K_G(I_G X)_{\mathbb{Q}}$ . In addition, Equation (27) means that the inertial Chern character homomorphism  $\tilde{\mathcal{C}h} : K_G(I_G X)_{\mathbb{Q}} \longrightarrow A_G^*(I_G X)_{\mathbb{Q}}$  preserves the augmented  $\psi$ -ring structures and factors through an isomorphism of augmented  $\psi$ -rings  $\hat{K}_G(I_G X)_{\mathbb{Q}} \rightarrow A_G^*(I_G X)_{\mathbb{Q}}$ . In particular, if  $G$  acts freely on  $X$ , then the inertial Chern character is an isomorphism of augmented  $\psi$ -rings.

## 6. $\lambda$ -POSITIVE ELEMENTS, THE INERTIAL DUAL, AND INERTIAL EULER CLASSES

Every  $\lambda$ -ring contains the semigroup of its  $\lambda$ -positive elements, which is an invariant of the  $\lambda$ -ring structure. In the case of ordinary equivariant K-theory, every class of a rank  $d$  vector bundle is a  $\lambda$ -positive element, although the converse need not be true. Nevertheless,  $\lambda$ -positive elements of degree  $d$  share many of the same properties as classes of rank  $d$ -vector bundles; for example, they have a top Chern class in Chow theory and an Euler class in K-theory. This is a consequence of the fact that the ordinary Chern character and Chern classes are compatible with the  $\lambda$  and  $\psi$ -ring structures.

In this section, we will introduce the framework to investigate the  $\lambda$ -positive elements of inertial K-theory for strongly Gorenstein inertial pairs. We will see that the  $\lambda$ -positive elements of degree  $d$  in inertial K-theory satisfy the inertial versions of these properties. We will also introduce a notion of duality for inertial K-theory which is necessary to define the inertial Euler class in inertial K-theory.

In a number of cases, e.g.  $\mathbb{P}(1, 2)$  and  $\mathbb{P}(1, 3)$ , we will see that the set of  $\lambda$ -positive elements yields integral structures on inertial K-theory and inertial Chow theory which will correspond, under a kind of mirror symmetry, to the usual integral structures on ordinary K-theory and Chow theory of an associated crepant resolution of the orbifold cotangent bundle.

**Remark 6.1.** All results in this section hold for possibly nondiagonalizable  $G$ , provided that  $K_G(I_G X)$  is replaced by the augmentation completion of its Gorenstein subring  $\hat{K}_G(I_G X)$ .

We begin by defining the appropriate notion of duality for inertial K-theory.

**Definition 6.2.** Consider the inertial K-theory  $(K_G(I_G X), \star, 1, \tilde{\epsilon}, \tilde{\psi})$  of a strongly Gorenstein pair  $(\mathcal{R}, \mathcal{S})$  associated to a proper action of a diagonalizable group  $G$  on  $X$ . The *inertial dual* is the map  $\tilde{\mathbf{D}} : K_G(I_G X) \longrightarrow K_G(I_G X)$  defined by

$$\tilde{\mathbf{D}}(\mathcal{V}) := \mathcal{V}^\dagger := \mathcal{V}^* \cdot \rho(\mathcal{S}^*),$$

where

$$\rho(\mathcal{F}) := (-1)^{\epsilon(\mathcal{F})} \det(\mathcal{F}^*) \quad (40)$$

for all classes of locally free sheaves  $\mathcal{F}$  in  $K_G(I_G X)$ , and  $\det(\mathcal{F}) = \lambda^{\epsilon(\mathcal{F})} \mathcal{F}$  is the class of the usual determinant line bundle of  $\mathcal{F}$ . Note that in this definition both  $\epsilon$  and  $\det$  are the usual, non-inertial forms.

**Theorem 6.3.** *Consider the inertial K-theory  $(K_G(I_G X), \star, 1, \tilde{\epsilon}, \tilde{\psi})$  of a strongly Gorenstein pair  $(\mathcal{R}, \mathcal{S})$  for a diagonalizable group  $G$  with a proper action on  $X$ .*

- (1)  $\tilde{\mathbf{D}}^2$  is the identity map, i.e.,  $\mathcal{F}^{\dagger\dagger} = \mathcal{F}$  for all  $\mathcal{F} \in K_G(I_G X)$ .
- (2) The inertial dual satisfies the equations

$$\tilde{\mathbf{D}} \circ \tilde{\epsilon} = \tilde{\epsilon} \circ \tilde{\mathbf{D}} = \tilde{\epsilon} \quad \text{and} \quad \tilde{\psi}^\ell \circ \tilde{\mathbf{D}} = \tilde{\mathbf{D}} \circ \tilde{\psi}^\ell \quad (41)$$

for all  $\ell \geq 1$ .

- (3) The inertial dual is a homomorphism of unital rings.

Before we give the proof of the theorem, we need to recall one fact from [FL85] about the ordinary dual in K-theory, and we need to prove a Riemann-Roch type of result for the ordinary dual.

**Lemma 6.4** ([FL85, I Lm 5.1]). *Let  $\mathcal{F}$  be any locally free sheaf of rank  $d$ . Then for all  $i$  with  $0 \leq i \leq m$  we have*

$$\lambda^i(\mathcal{F}) = \lambda^{d-i}(\mathcal{F}^*) \lambda^d(\mathcal{F}) \quad (42)$$

**Lemma 6.5** (Riemann-Roch for the Ordinary Dual). *Using the hypotheses and notation from Theorem 3.18, and using the definition of  $\rho$  given in Equation (40), we have*

$$(\iota_*(\mathcal{F}))^* = \iota_*(\rho(N_t^*) \cdot \mathcal{F}^*) \quad (43)$$

for all  $\mathcal{F}$  in  $K_G(Y)$ ,

*Proof.* We first observe, using Lemma 6.4, that for any locally free sheaf  $\mathcal{F} \in K_G(Y)$  we have

$$\lambda_{-1}(\mathcal{F})^* = \lambda_{-1}(\mathcal{F}) \rho(\mathcal{F}). \quad (44)$$

We also observe that ordinary dualization commutes with pullback and is a ring homomorphism. Because of these properties, the ordinary dual is a so-called *natural operation*, and the desired result follows immediately from K ock's ‘‘Riemann-Roch theorem without denominators’’ [K oc91, Satz 5.1].  $\square$

*Proof (of Theorem 6.3).* Part 1 follows from the identity  $\rho(\mathcal{F}^*) = (\rho(\mathcal{F}))^{-1}$ .

The first Equation of Part 2 follows from the definition of  $\tilde{\epsilon}$ . The second Equation of Part 2 follows from the identity  $\theta^n(\mathcal{S}) = \theta^n(\mathcal{S}^*)(\det(\mathcal{S}))^{\epsilon(\mathcal{S})-1}$  which follows from the splitting principle.

The proof of Part 3 is identical to the proof that  $\tilde{\psi}^n$  is a homomorphism for all  $n \geq 1$ , but where the Bott class  $\theta^n$  is replaced by the class  $\rho$  and the Adams-Riemann-Roch Theorem 3.18 is replaced by Lemma 6.5.  $\square$

**Definition 6.6.** Let  $(K, \cdot, 1, \lambda)$  be a  $\lambda$ -ring. For any integer  $d \geq 0$ , an element  $\mathcal{V} \in K$  is said to have  $\lambda$ -degree  $d$  if  $\lambda_t(\mathcal{V})$  is a degree- $d$  polynomial in  $t$ . The element  $\mathcal{V}$  is said to be a  $\lambda$ -positive element of degree  $d$  of  $K$  if it has  $\lambda$ -degree  $d$  for  $d \geq 1$  and  $\lambda^d(\mathcal{V})$  is a unit of  $K$ . A  $\lambda$ -positive element of degree 1 is said to be a  $\lambda$ -line element of  $K$ . Let  $\mathcal{P}_d := \mathcal{P}_d(K)$  be the set of  $\lambda$ -positive elements of degree  $d$  in  $K$ .

**Remark 6.7.** If the  $\lambda$ -ring  $(K, \cdot, 1, \lambda)$  has an involutive homomorphism  $K \longrightarrow K$  taking  $\mathcal{F} \mapsto \mathcal{F}^*$  which commutes with  $\lambda^i$  for all  $i \geq 0$  then it may be useful in the definition of a  $\lambda$ -positive element of degree  $d$  to assume, in addition, that  $(\lambda^d \mathcal{V})^{-1} = \lambda^d(\mathcal{V}^*)$ . However, we will later see that this condition is redundant for the virtual K-theory of  $B\mu_2$ ,  $\mathbb{P}(1, 2)$  and  $\mathbb{P}(1, 3)$ .

**Proposition 6.8.** *Let  $(K, \cdot, 1, \lambda)$  be a  $\lambda$ -ring.*

- (1) *Addition in  $K$  induces a map  $\mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \longrightarrow \mathcal{P}_{d_1+d_2}$  for all integers  $d_1, d_2 \geq 1$ .*
- (2) *Multiplication in  $K$  induces a map  $\mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \longrightarrow \mathcal{P}_{d_1 d_2}$  for all integers  $d_1, d_2 \geq 1$ . In particular, the set  $\mathcal{P}_1$  of  $\lambda$ -line elements of  $K$  forms a group.*
- (3) *An element  $\mathcal{L}$  in  $K$  is a  $\lambda$ -line element if and only if*

$$\psi^\ell(\mathcal{L}) = \mathcal{L}^\ell \quad (45)$$

*for all integers  $\ell \geq 1$ .*

- (4) *For all  $i, j \geq 1$ ,  $\lambda^i \circ \psi^j = \psi^j \circ \lambda^i$  in  $K$ .*
- (5) *If  $K$  is an augmented  $\lambda$ -ring with augmentation  $\epsilon$ , then for all integers  $i \geq 0$  and  $d \geq 1$ , we have  $\lambda^i : \mathcal{P}_d \longrightarrow \mathcal{P}_{\binom{d}{i}}$ . In particular,*

$$\epsilon(\lambda^i(\mathcal{V})) = \binom{d}{i}, \quad (46)$$

*where the right hand side is understood to be multiplied by the unit element 1 in  $K$ .*

*Proof.* Part 1 follows from the fact that the product of invertible elements is invertible. Part 2 follows from properties of the universal polynomials  $\mathbf{P}_n$  appearing in Part 5 of the definition of a  $\lambda$ -ring. Part 3 follows immediately from Equation (9). Part 4 follows from Equation (10).

To prove Part 5, we see that the properties of the universal polynomials  $\mathbf{P}_{m,n}$  (see Remark (3.1)) imply that  $\lambda^i : \mathcal{P}_d \longrightarrow \mathcal{P}_{\binom{d}{i}}$  for all  $i \geq 0$ .

To prove Equation (46), let us first suppose that  $\mathcal{F} := \mathcal{L}$  belongs to  $\mathcal{P}_1$ . Applying  $\epsilon$  to Equation (45) for  $\ell = 2$ , we obtain  $\epsilon(\psi^2(\mathcal{L})) = \epsilon(\mathcal{L}^2) = \epsilon(\mathcal{L})^2$ , but  $\epsilon(\psi^2(\mathcal{L})) = \epsilon(\mathcal{L})$ . Thus,  $\epsilon(\mathcal{L})^2 = \epsilon(\mathcal{L})$ , but since  $\mathcal{L}$  is invertible and  $\epsilon$  is a homomorphism of unital rings,  $\epsilon(\mathcal{L})$  is invertible. Therefore,  $\epsilon(\mathcal{L}) = 1$ . More generally, if  $\mathcal{F}$  belongs to  $\mathcal{P}_d$  for some integer  $d \geq 1$ , then Equation (13) implies that  $\epsilon(\binom{\mathcal{F}}{d}) = 1$ , since  $\lambda^d(\mathcal{F})$  belongs to  $\mathcal{P}_1$  by the properties of the universal polynomials  $\mathbf{P}_{m,n}$  in Part 6 of the definition of a  $\lambda$ -ring. Since  $\mathcal{F}$  has  $\lambda$ -degree  $d$ , we have

$$0 = \binom{\epsilon(\mathcal{F})}{d+1} = \binom{\epsilon(\mathcal{F})}{d} \frac{\epsilon(\mathcal{F}) - d}{d+1} = \frac{\epsilon(\mathcal{F}) - d}{d+1}.$$

Therefore,  $\epsilon(\mathcal{F}) = d$  and Equation (46) follows from Equation (13).  $\square$

In ordinary equivariant K-theory  $(K_G(X), \otimes, 1, \epsilon)$ , it is often useful to assume that  $[X/G]$  is connected. This is not an actual restriction, since  $K_G(X)$  can be expressed as the direct sum of  $(\lambda$  or  $\psi)$ -rings of the form  $K_G(U)$ , where  $[U/G]$  is a connected component of  $[X/G]$ . The condition that  $[X/G]$  is connected is equivalent to the condition that the image of the augmentation is  $\mathbb{Z}$  times the unit element 1, i.e., one may interpret the augmentation as a map  $\epsilon : K_G(X) \longrightarrow \mathbb{Z}$ .

For an inertial K-theory  $(K_G(IGX), \star, 1, \tilde{\epsilon})$ , an additional condition must be imposed in order for the inertial augmentation to have image equal to  $\mathbb{Z}$ .

**Definition 6.9.** Let  $X$  be an algebraic space with the action of  $G$ . We say that the action is *reduced* with respect to the inertial pair  $(\mathcal{R}, \mathcal{S})$  if  $\mathcal{S}_m = 0$  only if  $m = 1$ .

The following Proposition is immediate.

**Proposition 6.10.** *Consider the inertial K-theory  $(K_G(I_G X), \star, 1, \tilde{\epsilon})$  (respectively the rational inertial K-theory  $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon})$ ) for some inertial pair  $(\mathcal{R}, \mathcal{S})$ . The image of the inertial augmentation  $\tilde{\epsilon}$  is equal to  $\mathbb{Z}$  (respectively  $\mathbb{Q}$ ) times the unit element 1 of  $K_G(I_G X)$  if and only if  $[X/G]$  is connected and the action of  $G$  on  $X$  is reduced with respect to  $(\mathcal{R}, \mathcal{S})$ .*

In ordinary equivariant K-theory any vector bundle of rank  $d$  has  $\lambda$ -degree  $d$ . Thus, if  $[X/G]$  is connected, then by definition,  $(K_G(X), \cdot, 1, \epsilon, \lambda)$  (respectively  $(K_G(X)_{\mathbb{Q}}, \cdot, 1, \epsilon, \lambda)$ ) is generated as a group (respectively  $\mathbb{Q}$ -vector space) by the classes of vector bundles and hence by elements of  $\mathcal{P}$ .

In inertial K-theory  $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$ , the situation is more complicated. Equation (46) implies that if  $\mathcal{V}$  is in  $\mathcal{P}_d$ , then for any any connected component  $U$  of  $I_G X \setminus X^1$  which has  $\mathcal{S}$ -age equal to 0, the restriction  $\mathcal{V}|_U$ , must have ordinary rank equal to 0 on  $U$ . Therefore, the  $\mathbb{Q}$ -linear span of  $\mathcal{P}_d$  cannot be equal to  $K_G(I_G X)_{\mathbb{Q}}$ . Furthermore, even if  $[X/G]$  is connected and the action of  $G$  on  $X$  is reduced with respect to the inertial pair  $(\mathcal{R}, \mathcal{S})$ , there is no *a priori* reason that  $(K_G(I_G X)_{\mathbb{Q}}, \cdot, 1, \tilde{\epsilon}, \tilde{\lambda})$  is generated as a  $\mathbb{K}$ -vector space by its  $\lambda$ -positive elements for any field  $\mathbb{K}$  containing  $\mathbb{Q}$ .

This motivates the following definition.

**Definition 6.11.** Consider the inertial K-theory  $(K_G(I_G X)_{\mathbb{C}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$  of a strongly Gorenstein pair  $(\mathcal{R}, \mathcal{S})$  associated to a diagonalizable group  $G$  with a proper action on  $X$ . Let  $\mathcal{P}$  denote its  $\lambda$ -positive elements. Let  $\mathcal{K}_G(I_G X)_{\mathbb{Q}}$  be the  $\mathbb{Q}$ -vector space generated by the elements of  $\mathcal{P}$ . We call  $(\mathcal{K}_G(I_G X)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$  the *core subring of the inertial K-theory*.

**Corollary 6.12.**  $(\mathcal{K}_G(I_G X)_{\mathbb{Q}}, \star, 1, \tilde{\lambda})$  is a  $\lambda$ -subring of the inertial K-theory which is preserved by the inertial dual.

*Proof.* The proof follows from Part 2 and Part 4 of Proposition 6.8 and the fact that the inertial dual maps  $\mathcal{P}_d \mapsto \mathcal{P}_d$  for all  $d$ .  $\square$

One thing that makes the elements  $\mathcal{P}_d$  in  $(K_G(I_G X)_{\mathbb{Q}}, \cdot, 1, \tilde{\epsilon}, \tilde{\lambda})$  interesting is that in many ways they behave as though they were rank- $d$  vector bundles. In particular, they have inertial Euler classes in both K-theory and Chow rings.

**Proposition 6.13.** *Let  $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$  be the inertial K-theory of a strongly Gorenstein pair  $(\mathcal{R}, \mathcal{S})$  associated to a diagonalizable group  $G$  with a proper action on  $X$ .*

- (1) *The first inertial Chern class  $\tilde{c}^1 : \mathcal{P}_1 \longrightarrow A_G^{\{1\}}(I_G X)_{\mathbb{Q}}$  is a homomorphism of groups.*
- (2) *For all  $\mathcal{V}$  in  $\mathcal{P}_d$  and  $\mathcal{L}$  in  $\mathcal{P}_1$ ,*

$$\tilde{\gamma}_t(\mathcal{V} - d) = \sum_{i=0}^d t^i (1-t)^{d-i} \tilde{\lambda}^i(\mathcal{V}), \quad (47)$$

$$\tilde{\mathcal{C}h}(\mathcal{L}) = \widetilde{\exp}(\tilde{c}^1(\mathcal{L})), \quad (48)$$

and

$$\tilde{c}_t(\mathcal{V}) = \sum_{i=0}^d \tilde{c}^i(\mathcal{V}) t^i. \quad (49)$$

*Proof.* Part 1 follows from the fact that for all  $\mathcal{L}_1$  and  $\mathcal{L}_2$  in  $\mathcal{P}_1$ ,  $\widetilde{\mathcal{E}h}(\mathcal{L}_1 \star \mathcal{L}_2) = \widetilde{\mathcal{E}h}(\mathcal{L}_1) \star \widetilde{\mathcal{E}h}(\mathcal{L}_2)$ . Picking off terms in  $A_G^{\{1\}}(I_G X)_{\mathbb{Q}}$  and using  $\widetilde{\mathcal{E}h}^1 = \widetilde{c}^1$  and Equation (46) yields the desired result.

Equation (47) holds since for all  $\mathcal{V}$  in  $\mathcal{P}_d$ ,

$$\begin{aligned} \widetilde{\gamma}_t(\mathcal{V} - d) &= \frac{\widetilde{\lambda}_{\frac{t}{1-t}}(\mathcal{V})}{(1-t)^{-d}} \\ &= (1-t)^d \sum_{i=0}^d \left( \frac{t}{1-t} \right)^i \widetilde{\lambda}^i(\mathcal{V}) \\ &= \sum_{i=0}^d t^i (1-t)^{d-i} \widetilde{\lambda}^i(\mathcal{V}). \end{aligned}$$

Equation (48) follows from Equations (21) and (49), which yields

$$1 + t\widetilde{c}^1(\mathcal{L}) = \widetilde{\exp} \left( \sum_{n \geq 1} (-1)^{n-1} (n-1)! t^n \widetilde{\mathcal{E}h}^n(\mathcal{L}) \right),$$

which implies that  $\widetilde{\mathcal{E}h}^n(\mathcal{L}) = \widetilde{c}^1(\mathcal{L})^n / n!$ , as desired. Equation (49) follows from Equations (29) and (47).  $\square$

Ordinary equivariant K-theory also has an operation corresponding to taking the dual of a vector bundle. We now introduce an inertial version.

The inertial dual allows us to introduce a generalization of the classical notion of Euler class.

**Definition 6.14.** Let  $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \widetilde{c}, \widetilde{\lambda})$  be the inertial K-theory associated to the pair  $(\mathcal{R}, \mathcal{S})$ . Let  $\mathcal{V}$  belong to  $\mathcal{P}_d$ ; that is,  $\mathcal{V}$  is a  $\lambda$ -degree- $d$  element of  $K_G(I_G X)_{\mathbb{Q}}$ . The *inertial Euler class* in  $K_G(I_G X)_{\mathbb{Q}}$  of  $\mathcal{V}$  is defined to be

$$\widetilde{\lambda}_{-1}(\mathcal{V}^\dagger) = \sum_{i=0}^d (-1)^i \widetilde{\lambda}^i(\mathcal{V}^\dagger).$$

The *inertial Euler class* of  $\mathcal{V}$  in  $A_G^{\{d\}}(I_G X)_{\mathbb{Q}}$  is defined to be  $\widetilde{c}^d(\mathcal{V})$ .

The inertial Euler classes are multiplicative by Part 1 of Proposition 6.8 and the multiplicativity of  $\widetilde{c}_t$  and  $\widetilde{\lambda}_t$ .

Finally, we observe that  $\mathcal{P}_1$  is preserved by the action of certain groups. This will be useful in our analysis of the virtual K-theory of  $\mathbb{P}(1, n)$ .

**Definition 6.15.** Let  $(K, \cdot, 1, \psi, \epsilon)$  be an augmented  $\psi$ -ring. A *translation group* of  $K$  is an additive subgroup  $J$  of  $K$  such that for all  $n \geq 1$ ,  $j \in J$ , and  $x \in K$ , the following identities hold:

- (1)  $\psi^n(j) = nj$ ,
- (2)  $x \cdot j = \epsilon(x)j$ ,
- (3)  $\epsilon(K)J = J$ .

**Proposition 6.16.** Let  $(K, \cdot, 1, \psi, \epsilon)$  be an augmented  $\psi$ -ring. If  $J$  is a translation subgroup of  $K$ , then  $\epsilon(J) = 0$ ,  $J^2 = 0$ , and  $J$  is an ideal of the ring  $K$ . Furthermore,  $J$  acts freely on  $\mathcal{P}_1$ , where  $J \times \mathcal{P}_1 \longrightarrow \mathcal{P}_1$  is  $(j, \mathcal{L}) \mapsto j + \mathcal{L}$ .

*Proof.* For all  $j$  in  $J$  and integers  $n \geq 1$ ,  $\epsilon(\psi^n(j)) = \epsilon(j)$  by the definition of an augmented  $\psi$ -ring. On the other hand,  $\epsilon(\psi^n(j)) = \epsilon(nj) = n\epsilon(j)$  for all integers  $n \geq 1$  by Condition (1) in the definition of a translation group. Therefore,  $\epsilon(j) = 0$ . The fact that  $J^2 = 0$  and  $J$  is an ideal of  $K$  follows from Condition (2) in the definition of a translation group.

Consider  $\mathcal{L}$  in  $\mathcal{P}_1$  and  $j$  in  $J$ . We have

$$\psi^n(\mathcal{L} + j) = \psi^n(\mathcal{L}) + \psi^n(j) = \mathcal{L}^n + nj = (\mathcal{L} + j)^n,$$

where the second equality is by Equation (45) and Condition (1) in the definition of a translation group, and the last is from the binomial theorem and the fact that  $J^2 = 0$ . Also, notice that  $(\mathcal{L}^{-1} - j)(\mathcal{L} + j) = 1$ , so  $\mathcal{L} + j$  is invertible.  $\square$

## 7. EXAMPLES

In this section, we work out some examples of inertial  $\psi$ - and  $\lambda$ -rings.

**7.1. The classifying stack of a finite Abelian group.** In this section we discuss the case where  $X$  is a point with a trivial action by a finite group  $G$ . We begin with some general results and conclude with explicit computations for the special case of the cyclic group  $G = \mu_2$  of order 2.

**7.1.1. General results.** Let  $X$  be a point with the (trivial) action of a finite Abelian group  $G$ . The inertia scheme is  $I_G X = G$ , which has the action of  $G$  by conjugation. The orbifold K-theory of  $\mathcal{B}G := [X/G]$  is additively the Grothendieck group  $K_G(I_G X) = K_G(G)$  of  $G$ -equivariant vector bundles over  $G$ ; however, the orbifold product on  $K_G(G)$  differs from the ordinary one. The double inertia manifold is  $\mathbb{I}_G^2 X = G \times G$  with the diagonal conjugation action of  $G$ , the evaluation maps  $e_i : G \times G \longrightarrow G$  are the projection maps onto the  $i$ -th factor for  $i = 1, 2$ , and  $\mu : G \times G \longrightarrow G$  is the multiplication map. Let  $\mathcal{F}$  and  $\mathcal{G}$  be  $G$ -equivariant vector bundles on  $G$ , then  $\mathcal{F} \star \mathcal{G} := \mu_*(\mathcal{F} \boxtimes \mathcal{G})$  is the  $G$ -equivariant vector bundle over  $G$  whose fiber over the point  $m$  in  $G$  is

$$(\mathcal{F} \star \mathcal{G})_m = \bigoplus_{m_1 m_2 = m} \mathcal{F}_{m_1} \otimes \mathcal{G}_{m_2},$$

where the sum is over all pairs  $(m_1, m_2) \in G^2$  such that  $m_1 m_2 = m$ .

The orbifold K-theory  $(K_G(G), \star, \mathbf{1})$  of  $\mathcal{B}G$  can naturally be identified with the representation ring  $\text{Rep}(D(G))$  of the Drinfeld double  $D(G)$  of the group  $G$ . The ring  $\text{Rep}(D(G))$  has been studied in some detail in [Wit96].

In this case the orbifold Chern classes are all trivial, i.e.,  $\tilde{c}_t(\mathcal{F}) = \mathbf{1}$  for all  $\mathcal{F}$ , since  $\mathcal{C}h_t(\mathcal{F}) = \text{Ch}_t(\mathcal{F}) = \text{rk}(\mathcal{F})$  for every  $\mathcal{F} \in K_G(I_G X)$ .

Since  $\mathcal{S} = 0$  on  $I_G X$ , the orbifold  $\psi$ -ring  $(K_G(G), \star, \mathbf{1}, \tilde{\psi})$  structure agrees with the ordinary  $\psi$ -ring structure, i.e.,  $\tilde{\psi}^i := \psi^i$  for all  $i \geq 1$ . If  $r$  is the exponent of  $G$  (i.e., the least common multiple of the orders of elements of  $G$ ), then  $\psi^{j+r} = \psi^j$  for all  $j \geq 1$ .

**7.1.2. The classifying stack  $\mathcal{B}\mu_2$ .** We now consider the special case where  $G = \mu_2$  is the cyclic group of order 2. For each  $m \in G$  and each irreducible representation  $\alpha \in \text{Irrep}(\mu_2) = \{\pm 1\}$ , let  $V_m^\alpha$  denote the bundle on  $G$  which is 0 away from the one-point set  $\{m\} \in I_G X = \mu_2$  and which is equal to  $\alpha$  on  $\{m\}$ . In this case the free Abelian group  $K_{\mu_2}(\mu_2)$  decomposes as  $K_{\mu_2}(\mu_2) = K_{\mu_2}(\{1\}) \oplus K_{\mu_2}(\{-1\})$  and has a basis consisting of the four elements  $V_1^1, V_1^{-1}, V_{-1}^1, V_{-1}^{-1}$ .

**Proposition 7.1.** *The orbifold  $\lambda$ -ring  $(K_{\mu_2}(\mu_2)_{\mathbb{Q}}, \star, \mathbf{1}, \tilde{\lambda})$  satisfies*

$$\tilde{\lambda}_t(V_1^1) = \mathbf{1} + t, \quad (50)$$

$$\tilde{\lambda}_t(V_1^{-1}) = \mathbf{1} + tV_1^{-1}, \quad (51)$$

$$\tilde{\lambda}_t(V_{-1}^1) = \mathbf{1} + tV_{-1}^1 + \frac{t^2}{2(1+t)}(1 - V_{-1}^1), \quad (52)$$

and

$$\tilde{\lambda}_t(V_{-1}^{-1}) = 1 + V_{-1}^{-1}t + \frac{t^2}{2(1-t^2)}(1 - tV_1^{-1} - V_{-1}^1 + tV_{-1}^{-1}). \quad (53)$$

There are four elements in  $\mathcal{P}_1$ , namely,  $V_1^{\pm 1}$  and

$$\sigma_{\pm} := \frac{1}{2}(V_1^1 + V_1^{-1} \pm (V_{-1}^1 - V_{-1}^{-1})),$$

with the orbifold multiplication given by  $\sigma_{\pm} \star \sigma_{\pm} = V_1^1$ , and  $V_1^{-1} \star \sigma_{\pm} = \sigma_{\mp}$ , and  $\sigma_+ \star \sigma_- = V_1^{-1}$ . Furthermore, the core subring  $\mathcal{K}_{\mu_2}(\mu_2)_{\mathbb{Q}}$  of inertial K-theory (see Definition 6.11) is the  $\mathbb{Q}$ -subalgebra of  $K_{\mu_2}(\mu_2)_{\mathbb{Q}}$  generated by  $\{V_1^{-1}, \sigma_+\}$ .

*Proof.* The positivity of  $V_1^{\pm 1}$  is immediate since the orbifold  $\lambda$ -ring structure reduces to the ordinary  $\lambda$ -ring structure on the untwisted sector. The other two elements follow from performing a direct calculation using Equation (45).  $\square$

Since the action of  $\mu_2$  on a point is not reduced, the  $\mathbb{Q}$ -span of  $\mathcal{P}$  cannot be equal to  $K_{\mu_2}(\mu_2)_{\mathbb{Q}}$ .

**7.2. The virtual K-theory and virtual Chow ring of  $\mathbb{P}(1, n)$ .** Let  $X := \mathbb{C}^n \setminus \{0\}$  and  $G := \mathbb{C}^{\times}$  with the action  $G \times X \longrightarrow X$  taking  $(t, (x, y)) \mapsto (tx, t^n y)$ . In this section we first develop some general results about the virtual K-theory and virtual Chow theory of the weighted projective line  $\mathbb{P}(1, n) := [X/G]$ . We then work out the full theory in the case of  $\mathbb{P}(1, 2)$  and compare our results to the usual K-theory and Chow of the resolution of singularities for the cotangent bundle  $T^*\mathbb{P}(1, 2)$ . Finally, we do the same for the case of  $\mathbb{P}(1, 3)$ .

**7.2.1. General results.**

*Notation 7.2.* Let  $\zeta_n := \exp(2\pi i/n)$ . We will use additive notation for the group  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ , which we will regard as a subgroup of  $\mathbb{C}^{\times}$  via the inclusion  $m \mapsto \zeta_n^m$ . Similarly, we will denote the component of the inertia variety  $X^{\zeta_n^m}$  by  $X^m$  for all  $m \in \{0, \dots, n-1\}$ .

The inertia variety is  $I_G X = \coprod_{m=0}^{n-1} X^m$ . With respect to the ordinary product, we have the isomorphism of rings

$$K_G(X^0) \cong \frac{\mathbb{Z}[\chi_0, \chi_0^{-1}]}{\langle (\chi_0 - 1)(\chi_0^n - 1) \rangle}, \quad (54)$$

and for  $m \in \{1, \dots, n-1\}$ , we have

$$K_G(X^m) \cong \frac{\mathbb{Z}[\chi_m, \chi_m^{-1}]}{\langle \chi_m^n - 1 \rangle}. \quad (55)$$

Let  $d_m$  denote the rank of the free  $\mathbb{Z}$ -module  $K_G(X^m)$ , i.e.,  $d_0 = n+1$  and  $d_m = n$  for  $m \in \{1, 2, \dots, n-1\}$ . Let  $y_m^a$  be the class of  $\chi_m^a$  in  $K_G(X^m)$  for all  $m \in \{0, 1, \dots, n-1\}$  and  $a = 0, \dots, d_m - 1$ , then  $\{y_m^a\}$  is a basis for  $K(I\mathbb{P}(1, n)) = K_G(I_G X)$ . Therefore,  $K(I\mathbb{P}(1, n))$  is a free  $\mathbb{Z}$ -module of rank  $n^2 + 1$ .

We need to calculate the inertial pair  $(\mathcal{R}, \mathcal{S})$  for virtual K-theory. We have  $\mathcal{S} = y_m^1$ ; hence, the  $\mathcal{S}$ -age of  $X^m$  is 1 if  $m \neq 0$ . We recall from [EJK12] that in virtual K-theory we have

$$\mathcal{R} = \mathbb{T}|_{\mathbb{I}_G^2 X} + \mathbb{T}|_{\mathbb{I}_G^2 X} - e_1^* \mathbb{T}|_{I_G X} - e_2^* \mathbb{T}|_{I_G X}. \quad (56)$$

If  $m_1 = 0$ , then  $\mathcal{R}|_{X^{0, m_2}} = 0$  for all  $m_2 \in \{0, 1, \dots, n-1\}$  (and similarly if  $m_2 = 0$ ). Otherwise, under the isomorphism  $e_1 : X^{m_1, m_2} \longrightarrow X^{m_1}$ ,  $\mathcal{R}|_{X^{m_1, m_2}}$  can be identified with  $(y_{m_1}^1 + y_{m_1}^n) + y_{m_1}^n - y_{m_1}^n - y_{m_1}^n = y_{m_1}^1$  by Equation (56). Hence, we have

$$y_{m_1}^{a_1} \star_{\text{eu } \mathbb{T}^{\text{virt}}} y_{m_2}^{a_2} = y_{m_1+m_2}^{a_1+a_2} \cdot \text{eu}(\mathcal{S}_{m_1} + \mathcal{S}_{m_2} - \mathcal{S}_{m_1+m_2}),$$

where the products on the right hand side are the ordinary ones and

$$\text{eu}(\mathcal{S}_{m_1} + \mathcal{S}_{m_2} - \mathcal{S}_{m_1+m_2}) = \begin{cases} 0 & \text{if either } m_1 = 0 \text{ or } m_2 = 0, \\ y_{m_1+m_2}^0 - 2y_{m_1+m_2}^{-1} + y_{m_1+m_2}^{-2} & \text{if } m_1 + m_2 = n, m_1 m_2 \neq 0, \\ y_{m_1+m_2}^0 - y_{m_1+m_2}^{-1} & \text{otherwise.} \end{cases}$$

The virtual augmentation  $\tilde{\epsilon} : K(\mathbb{I}\mathbb{P}(1, n)) \longrightarrow K(\mathbb{I}\mathbb{P}(1, n))$  satisfies

$$\tilde{\epsilon}(y_0^a) = y_0^0,$$

and for  $m \in \{1, \dots, n-1\}$  we have

$$\tilde{\epsilon}(y_m^a) = 0$$

for all  $a \in \mathbb{Z}$ .

In order to calculate the virtual  $\psi$  operations, for all  $m \in \{1, \dots, n-1\}$  we need the  $\ell$ -th Bott class  $\theta^\ell(\mathcal{S}_m^*)$  in  $K_G(X^m)$ , which satisfies

$$\theta^\ell(\mathcal{S}_m^*) = \theta^\ell(y_m^{-1}) = \sum_{i=0}^{\ell-1} y_m^{-i}.$$

This yields the virtual  $\psi$  operations  $\tilde{\psi}^k : K(\mathbb{I}\mathbb{P}(1, n)) \longrightarrow K(\mathbb{I}\mathbb{P}(1, n))$ .

Let us now calculate the virtual inertial Chow ring of  $X$ . With respect to the ordinary product, we have the isomorphism of rings

$$A_G^*(X^0) \cong \frac{\mathbb{Z}[q_0]}{\langle nq_0^2 \rangle}, \quad (57)$$

and, for  $m \in \{1, \dots, n-1\}$  we have

$$A_G^*(X^m) \cong \frac{\mathbb{Z}[q_m]}{\langle nq_m \rangle}, \quad (58)$$

where  $q_m$  has ordinary Chow grading 1 for all  $m$ . Let  $c_m^a$  be the class of  $q_m^a$  for all  $m \in \{0, \dots, n-1\}$  and  $a \geq 0$ . The inertial grading of  $c_0^a$  is  $a$  and of  $c_m^a$  is  $a+1$  for  $m \in \{1, \dots, n-1\}$ .

The multiplication table for the virtual inertial Chow ring is given by

$$c_{m_1}^{a_1} \star_{\text{virt}} c_{m_2}^{a_2} = c_{m_1+m_2}^{a_1+a_2} \text{eu}(\mathcal{S}_{m_1} + \mathcal{S}_{m_2} - \mathcal{S}_{m_1+m_2}),$$

where we have the following identity in  $A^*(\mathbb{I}\mathbb{P}(1, n))$ :

$$\text{eu}(\mathcal{S}_{m_1} + \mathcal{S}_{m_2} - \mathcal{S}_{m_1+m_2}) = \begin{cases} 1 & \text{if either } m_1 = 0 \text{ or } m_2 = 0, \\ c_{m_1+m_2}^2 & \text{if } m_1 + m_2 = n, m_1 m_2 \neq 0, \\ c_{m_1+m_2}^1 & \text{otherwise.} \end{cases}$$

This multiplication is simple on  $A^*(IP(1, n))_{\mathbb{Q}}$  since  $c_0^0$  is the identity element, while the product of non-identity elements  $c_{m_1}^{a_1}$  and  $c_{m_2}^{a_2}$  is 0 by the inertial grading.

The virtual Chern character homomorphism  $\mathcal{Ch} : K(IP(1, n)) \longrightarrow A^*(IP(1, n))_{\mathbb{Q}}$  satisfies, for all  $a \geq 0$ ,

$$\widetilde{\mathcal{Ch}}(y_0^a) = c_0^0 + ac_0^1, \quad (59)$$

and, for  $m \in \{1, \dots, n-1\}$  we have

$$\widetilde{\mathcal{Ch}}(y_m^a) = c_m^0. \quad (60)$$

We now show that the inertial  $\psi$  operations are almost  $n$ -periodic.

**Definition 7.3.** Let  $\mathbb{K}$  be  $\mathbb{Q}$  or  $\mathbb{C}$ . For all  $m \in \{1, \dots, n-1\}$ , define elements  $\Delta_m = \sum_{i=0}^{n-1} y_m^i$  in  $K_G(X^m)$  (respectively  $K_G(X^m)_{\mathbb{K}}$ ) and  $\Delta_0 = y_0^0 - y_0^n$  in  $K_G(X^0)$  (respectively  $K_G(X^0)_{\mathbb{K}}$ ). Let  $J$  (respectively  $J_{\mathbb{K}}$ ) be the additive group (respectively  $\mathbb{K}$ -vector space) generated by  $\{\Delta_i\}_{i=0}^n$ .

For all  $k \geq 1$ , let  $\widetilde{\psi}_m^k(\mathcal{F}) := \psi^k(\mathcal{F}_m)$  for all  $\mathcal{F} = \sum_{m=0}^n \mathcal{F}_m$ , where  $\mathcal{F}_m$  belongs to  $K_G(X^m)$ .

Let  $\varphi_0 : K(IP(1, n)) \longrightarrow \mathbb{Z}$  be the additive map that is supported on  $K_G(X^0)$  such that  $\varphi_0(y_0^s) = s$  for all  $s \in \{0, \dots, n\}$ .

Finally, let  $\widetilde{\psi}^0$  be the inertial augmentation  $\widetilde{\epsilon}$ .

**Lemma 7.4.** Let  $(K(IP(1, n)), \star, 1, \widetilde{\epsilon}, \widetilde{\psi})$  be the virtual  $K$ -theory ring.

(1) For all  $m \in \{0, \dots, n-1\}$  and  $\mathcal{F}_m$  in  $K_G(X^m)$ , we have the identity with respect to the ordinary product

$$\Delta_m \cdot \mathcal{F}_m = \epsilon_m(\mathcal{F}_m) \Delta_m. \quad (61)$$

(2) For all  $j$  in  $J$  and  $\mathcal{F}$  in the virtual  $K$ -theory ring  $K(IP(1, n))$ ,

$$\mathcal{F} \star j = \widetilde{\epsilon}(\mathcal{F})j, \quad (62)$$

$J \star J = 0$  and  $\widetilde{\epsilon}(J) = 0$ .

(3) For all  $\ell \geq 1$  and  $j \in J$ , we have the identity

$$\widetilde{\psi}^\ell(j) = \ell j. \quad (63)$$

In particular,  $J$  is a translation group of the virtual  $K$ -theory  $K(IP(1, n))$ .

*Proof.* Equation (61) follows from the identity  $(y_0^n - 1)(y_0^1 - 1) = 0$  in  $K_G(X^0)$ , and  $y_m^n - 1 = 0$  in  $K_G(X^m)$  for all  $m \neq 0$ .

Equation (62) follows immediately from the definition of  $\star$  and Equation (61). The fact that  $J \star J = 0$  follows from Equation (62) and the fact that  $\widetilde{\epsilon}(\Delta_m) = 0$  for all  $m$ .

To prove equation (63), we first consider

$$\begin{aligned} \widetilde{\psi}^\ell(\Delta_0) &= \psi^\ell(1 - y_0^n) = 1 - y_0^{n\ell} \\ &= 1 - (1 + (y_0^n - 1))^\ell \\ &= 1 - (1 + \ell(y_0^n - 1)) = \ell \Delta_0, \end{aligned}$$

where we have used the binomial series and the relation  $(y_0^n - 1)(y_0^1 - 1) = 0$  in the fourth equality. Suppose  $m \neq 0$  and  $x = y_m^1$ . We have the following, where all

products are understood to be ordinary products,

$$\begin{aligned}
\tilde{\psi}^\ell(\Delta_m) &= \psi^\ell(\Delta_m) \cdot \theta^\ell(x^{-1}) = \psi^\ell\left(\sum_{i=0}^{n-1} x^i\right) \sum_{j=0}^{\ell-1} (x^{-j}) \\
&= \sum_{i=0}^{n-1} (x^\ell)^i \sum_{j=0}^{\ell-1} x^{-j} = \frac{1 - (x^\ell)^n}{1 - x^\ell} \frac{1 - (x^{-1})^\ell}{1 - x^{-1}} x^n = \frac{1 - x^{\ell n}}{1 - x^\ell} \frac{1 - x^\ell}{1 - x} x^{n-\ell+1} \\
&= \frac{1 - x^{\ell n}}{1 - x} x^{n-\ell+1} = \frac{1 - (1 + (x^n - 1))^\ell}{1 - x} x^{n-\ell+1} \\
&= \frac{1 - (1 + \ell(x^n - 1))}{1 - x} x^{n-\ell+1} = \ell \frac{1 - x^n}{1 - x} = \ell \Delta_m,
\end{aligned}$$

where we have used the relation  $x^n - 1$  in the fourth equality and have used the binomial expansion and the relation  $x^n - 1$  in the eighth.

This establishes Equation (63).  $\square$

**Proposition 7.5.** *For all  $k \geq 0$  and  $a \in \{0, \dots, n-1\}$ , we have the identity in virtual  $K$ -theory  $(K(I\mathbb{P}(1, n)), \star, 1, \tilde{\epsilon}, \tilde{\psi})$*

$$\tilde{\psi}^{nk+a} = \tilde{\psi}^a + k\Delta_0\varphi_0 + \sum_{m=1}^n k\Delta_m\epsilon_m, \quad (64)$$

where  $\epsilon_m(\mathcal{F})$  denotes the ordinary augmentation of the summand  $\mathcal{F}_m$  in  $K_G(X^m)$  of  $\mathcal{F}$ .

*Proof.* If  $a \in \{0, \dots, n-1\}$ ,  $k \geq 0$ ,  $s \in \{0, \dots, n\}$  and  $x = y_0^1$ , then

$$\begin{aligned}
\tilde{\psi}_0^{nk+a}(x^s) &= (x^{ns})^k x^{as} \\
&= (1 + (x^{ns} - 1))^k x^{as} \\
&= (1 + k(x^{ns} - 1))x^{as} \\
&= x^{sa} + kx^{sa}((1 + (x^n - 1))^s - 1) \\
&= x^{sa} + kx^{sa}(1 + s(x^n - 1) - 1) \\
&= x^{sa} + ksx^{sa}(x^n - 1) \\
&= x^{sa} + ks(1 + (x^{sa} - 1))(x^n - 1) \\
&= x^{sa} + ks(x^n - 1) + ks(x^{sa} - 1)(x^n - 1) \\
&= x^{sa} + ks(x^n - 1) = x^{sa} + ks\Delta_0,
\end{aligned}$$

where we have used the relation  $(x^n - 1)(x - 1) = 0$  in  $K_G(X^0)$  in the third, fifth, and ninth equalities. Therefore, for all  $n, k \geq 0$  and  $a \in \{0, \dots, n-1\}$ , we have

$$\tilde{\psi}_0^{nk+a} = \tilde{\psi}_0^a + k\Delta_0\varphi_0. \quad (65)$$

If  $m \in \{1, \dots, n-1\}$ , then, adopting the convention that  $\theta^0(0) = 1$  and  $\theta^0(y_m^s) = 0$  for all  $s$ , we obtain

$$\begin{aligned}
\tilde{\psi}_m^{nk+a}(y_m^s) &= \psi_m^{nk+a}(y_m^s)\theta^{nk+a}(\mathcal{S}_m^*) \\
&= \psi_m^a(y_m^s)(k\Delta_m + \theta^a(\mathcal{S}_m^*)) \\
&= k\psi_m^a(y_m^s)\Delta_m + \psi_m^a(y_m^s)\theta^a(\mathcal{S}_m^*) \\
&= k\epsilon_m(\psi_m^a(y_m^s))\Delta_m + \tilde{\psi}_m^a(y_m^s) \\
&= k\Delta_m + \tilde{\psi}_m^a(y_m^s),
\end{aligned}$$

where we have used  $\mathcal{S}_m = y_m^1$  for all  $m \in \{1, \dots, n-1\}$ , the relation, with respect to the ordinary multiplication,  $(y_m^1)^n - 1 = 0$  in  $K_G(X^m)$ , Equation (61), and that  $\epsilon_m \psi_m^a = \epsilon_m$ . Consequently, we have

$$\tilde{\psi}_m^{nk+a} = \tilde{\psi}_m^a + k\Delta_m \epsilon_m \quad (66)$$

for all  $n, k \geq 0$ ,  $a \in \{0, \dots, n-1\}$ , and  $m \in \{1, \dots, n-1\}$ .

Equations (65) and (66) yield Equation (64).  $\square$

**Proposition 7.6.** *An element  $\mathcal{L}$  in virtual K-theory  $(K(\mathbb{IP}(1, n))_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\psi})$  is a  $\lambda$ -line element with respect to its inertial  $\lambda$ -ring structure if and only if  $\tilde{\epsilon}(\mathcal{L}) = 1$  and Equation (45) holds for all  $\ell \in \{1, \dots, n-1\}$ .*

*Proof.* First of all, notice that Equation (45) trivially holds for  $\ell = 1$  by definition of a  $\psi$ -ring. Suppose that  $\mathcal{L}$  in  $K(\mathbb{IP}(1, n))_{\mathbb{Q}}$  satisfies Equation (45) for all  $\ell \in \{1, \dots, n\}$ . We now prove that Equation (45) holds for all  $\ell$  by induction. Suppose that Equation (45) holds for all  $\ell \in \{1, \dots, nk+a\}$  for some  $a \in \{0, \dots, n-1\}$  and  $k \geq 0$ , then Equation (64) implies that

$$\tilde{\psi}^{n(k+1)+a}(\mathcal{L}) = \tilde{\psi}^a(\mathcal{L}) + (k+1)j(\mathcal{L}), \quad (67)$$

where  $j(\mathcal{L}) := \varphi_0(\mathcal{L})\Delta_0 + \sum_{m=1}^n \Delta_m \epsilon_m(\mathcal{L})$  belongs to the  $J$ . However,

$$\begin{aligned} \mathcal{L}^{n(k+1)+a} &= \mathcal{L}^{nk+a} \mathcal{L}^n \\ &= (\tilde{\psi}^a(\mathcal{L}) + kj(\mathcal{L}))(\tilde{\psi}^0(\mathcal{L}) + j(\mathcal{L})) \\ &= (\tilde{\psi}^a(\mathcal{L}) + kj(\mathcal{L}))(1 + j(\mathcal{L})) \\ &= \tilde{\psi}^a(\mathcal{L}) + kj(\mathcal{L}) + \tilde{\psi}^a(\mathcal{L})j(\mathcal{L}) + kj(\mathcal{L})^2 \\ &= \tilde{\psi}^a(\mathcal{L}) + kj(\mathcal{L}) + kj(\mathcal{L}) \\ &= \tilde{\psi}^a(\mathcal{L}) + (k+1)j(\mathcal{L}) \\ &= \tilde{\psi}^{n(k+1)+a}(\mathcal{L}), \end{aligned}$$

where we have used the induction hypothesis and Equation (67) in the second equality, the definition  $\tilde{\psi}^0 = \tilde{\epsilon}$  in the third equality, Lemma (7.4) in the fifth, the fact that  $\tilde{\epsilon} \circ \tilde{\psi}^q = \tilde{\epsilon}$  in the sixth, and Equation (67) in the seventh.  $\square$

**Remark 7.7.** Proposition (7.6) reduces the problem of finding  $\lambda$ -line elements of  $K(\mathbb{IP}(1, n))_{\mathbb{Q}}$  to solving a finite number of equations for  $n^2 + 1$  (the rank of  $K(\mathbb{IP}(1, n))$ ) unknowns. Furthermore, since the action of the translation group  $J$ , which is rank  $n$ , respects  $\mathcal{P}_1$  by Proposition (6.16), it is enough to solve for only  $n^2 - n + 1$  variables satisfying Equation (45) for all  $\ell \in \{0, \dots, n-1\}$ , as all other  $\lambda$ -line elements will be their  $J$  translates. Finally, we will see that in order to compare the virtual K-theory of  $\mathbb{P}(1, n)$  with the K-theory of a toric resolution of  $\mathbb{P}(1, n)$ , we will need to find the  $\lambda$ -line elements in the  $\mathbb{C}$ -algebra  $K(\mathbb{IP}(1, n))_{\mathbb{C}}$ , in general.

**Corollary 7.8.** *Let  $\mathcal{P}_1$  be the semigroup of  $\lambda$ -line elements of the virtual K-theory  $(K(\mathbb{IP}(1, n))_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$ . Each  $J_{\mathbb{Q}}$ -orbit in  $\mathcal{P}_1$  contains a unique representative  $\mathcal{L}$  such that  $\mathcal{L}^{\star n} = 1$ .*

*Proof.* Given  $\mathcal{F}$  in  $\mathcal{P}_1$ , we have  $\mathcal{F}^{\star n} = \tilde{\psi}^n(\mathcal{F}) = 1 + j$  for some  $j$  in  $J_{\mathbb{Q}}$  by Proposition (7.5). If  $\mathcal{L} = \mathcal{F} - \frac{1}{n}(j)$ , then  $\mathcal{L}^{\star n} = (\mathcal{F} + \frac{j}{n})^{\star n} = \mathcal{F}^{\star n} - j = 1 + j - j = 1$ .  $\square$

7.2.2. *The Virtual K-theory and virtual Chow ring of  $\mathbb{P}(1, 2)$ .* We now study the virtual K-theory and virtual Chow theory of the weighted projective line  $\mathbb{P}(1, 2) := [X/G]$  which, in this case, are isomorphic to the orbifold K-theory and orbifold Chow theory, respectively, of the cotangent bundle  $T^*\mathbb{P}(1, 2)$ .

**Remark 7.9.** For the remainder of this section, unless otherwise specified, all products are with respect to the virtual products.

Let  $\tilde{\lambda} : K(I\mathbb{P}(1, 2))_{\mathbb{Q}} \longrightarrow K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$  denote the induced virtual  $\lambda$ -ring structure. In order to describe the group of  $\lambda$ -line elements  $\mathcal{P}_1$  of  $(K(I\mathbb{P}(1, 2))_{\mathbb{Q}}, \cdot, 1, \tilde{\lambda})$ , it will be useful to introduce the injective map  $f : \mathbb{Q}^2 \longrightarrow K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$  defined by

$$f(\alpha, \beta) := -\alpha\Delta_0 + \beta\Delta_1, \quad (68)$$

whose image is the translation group  $J_{\mathbb{Q}}$  of  $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ .

Consider the following injective maps  $\mathbb{Q}^2 \longrightarrow K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ :

$$\rho_0(\alpha, \beta) := y_0^0 + f(\alpha, \beta), \quad (69)$$

$$\rho_1(\alpha, \beta) := y_0^1 + f(\alpha, \beta), \quad (70)$$

and

$$\rho_{\pm}(\alpha, \beta) := \frac{1}{2}(y_0^0 + y_0^1 \pm y_1^0) + f(\alpha, \beta). \quad (71)$$

**Proposition 7.10.** *The group of  $\lambda$ -line elements  $\mathcal{P}_1$  of the virtual K-theory  $(K(I\mathbb{P}(1, 2))_{\mathbb{Q}}, \star_{\text{virt}}, 1, \tilde{\lambda})$  is the disjoint union of the images of the four maps  $\rho_0, \rho_1, \rho_{\pm}$  and the restriction of the inertial dual  $\mathcal{P}_1 \longrightarrow \mathcal{P}_1$  agrees with the operation of taking the inverse. In particular,  $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$  is spanned as a  $\mathbb{Q}$ -vector space by  $\mathcal{P}_1$  and, therefore, the virtual K-theory is equal to its core subring. The multiplication in  $\mathcal{P}_1$  is*

$$\begin{aligned} \rho_0(\alpha, \beta)\rho_0(\alpha', \beta') &= \rho_0(\alpha + \alpha', \beta + \beta'), \\ \rho_0(\alpha, \beta)\rho_1(\alpha', \beta') &= \rho_1(\alpha + \alpha', \beta + \beta'), \\ \rho_0(\alpha, \beta)\rho_{\pm}(\alpha', \beta') &= \rho_{\pm}(\alpha + \alpha', \beta + \beta'), \\ \rho_1(\alpha, \beta)\rho_1(\alpha', \beta') &= \rho_0(\alpha + \alpha' + 1, \beta + \beta'), \\ \rho_1(\alpha, \beta)\rho_{\pm}(\alpha', \beta') &= \rho_{\mp}(\alpha + \alpha' + \frac{1}{2}, \beta + \beta' \pm \frac{1}{2}), \\ \rho_{\pm}(\alpha, \beta)\rho_{\pm}(\alpha', \beta') &= \rho_0(\alpha + \alpha' + \frac{1}{2}, \beta + \beta' \pm \frac{1}{2}), \end{aligned}$$

and

$$\rho_+(\alpha, \beta)\rho_-(\alpha', \beta') = \rho_1(\alpha + \alpha', \beta + \beta').$$

The inverses are given by

$$\begin{aligned} \rho_0(\alpha, \beta)^{-1} &= \rho_0(-\alpha, -\beta), \\ \rho_1(\alpha, \beta)^{-1} &= \rho_1(-(1 + \alpha), -\beta), \end{aligned}$$

and

$$\rho_{\pm}(\alpha, \beta)^{-1} = \rho_{\pm}(-(\alpha + \frac{1}{2}), -\beta \mp \frac{1}{2}).$$

*Proof.* The elements of  $\mathcal{P}_1$  are determined by solving Equation (45) through a direct calculation.

To prove that  $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$  is spanned as a  $\mathbb{Q}$ -vector space by the elements of  $\mathcal{P}_1$ , it is clear that the following 5 vectors span the  $\mathbb{Q}$ -space  $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ :  $\rho_0(0, 0) = y_0^0$ ,  $\rho_0(1, 0) = y_0^2$ ,  $\rho_1(0, 0) = y_0^1$ ,  $\rho_{\pm}(0, 0) = \frac{1}{2}(y_0^0 + y_0^1 \pm y_1^0)$ .  $\square$

**Remark 7.11.** There is an isomorphism of groups  $\mathcal{P}_1 \longrightarrow (\mathbb{Z}_2)^2 \rtimes J_{\mathbb{Q}}$ , where  $\rho_+(-\frac{1}{4}, -\frac{1}{4}) + j \mapsto ((1, 0), j)$   $\rho_-(-\frac{1}{4}, \frac{1}{4}) + j \mapsto ((0, 1), j)$  for all  $j$  in  $J_{\mathbb{Q}}$ . Here, the generators  $\rho_{\pm}(-\frac{1}{4}, \mp\frac{1}{4})$  are the unique representatives of the corresponding  $J_{\mathbb{Q}}$ -orbits from Corollary (7.8).

A direct calculation yields the following.

**Proposition 7.12.** *The inertial first Chern class for virtual K-theory is a homomorphism of groups  $\tilde{c}^1 : \mathcal{P}_1 \longrightarrow A^{\{1\}}(IP(1, 2))_{\mathbb{Q}}$ , where*

$$\begin{aligned}\tilde{c}^1(\rho_0(\alpha, \beta)) &= 2\alpha c_0^1 + 2\beta c_1^0, \\ \tilde{c}^1(\rho_1(\alpha, \beta)) &= (2\alpha + 1)c_0^1 + 2\beta c_1^0, \\ \tilde{c}^1(\rho_{\pm}(\alpha, \beta)) &= (2\alpha + \frac{1}{2})c_0^1 + (2\beta \pm \frac{1}{2})c_1^0.\end{aligned}$$

The virtual K-theory ring has a particularly simple form in terms of these  $\lambda$ -line elements.

**Proposition 7.13.** *Let  $(K(IP(1, 2))_{\mathbb{Q}}, \star_{virt}, 1 := y_0^0)$  be the virtual K-theory ring. We have two isomorphisms of  $\mathbb{Q}$ -algebras (and  $\psi$ -rings)*

$$\Phi_{\pm} : \frac{\mathbb{Q}[\sigma, \sigma^{-1}, \tau, \tau^{-1}]}{\langle (\tau - 1)(\tau^2 - 1), (\sigma - 1)(\sigma^2 - 1), (\sigma - \tau)(\tau - 1) \rangle} \longrightarrow K(IP(1, 2))_{\mathbb{Q}}, \quad (72)$$

where  $\Phi_{\pm}(\tau) := \rho_{\pm}(0, 0) = \frac{1}{2}(y_0 + y_0^1 \pm y_1^0)$  and  $\Phi_{\pm}(\sigma) := \rho_1(0, 0) = y_0^1$ . Similarly, we have two isomorphisms of graded  $\mathbb{Q}$ -algebras

$$\Psi_{\pm} : \frac{\mathbb{Q}[\mu, \nu]}{\langle \mu, \nu \rangle^2} \longrightarrow A^*(IP(1, 2))_{\mathbb{Q}}, \quad (73)$$

where  $\mu, \nu$  in  $A^{\{1\}}(IP(1, 2))_{\mathbb{Q}}$  and  $\Psi_{\pm}(\nu) := \tilde{c}^1(\rho_{\pm}(0, 0)) = \frac{1}{2}(c_0^1 \pm c_1^0)$  and  $\Psi_{\pm}(\mu) := \tilde{c}^1(\rho_1(0, 0)) = c_0^1$ . Under the identifications  $\Phi_{\pm}$  and  $\Psi_{\pm}$ , the inertial Chern character  $\mathcal{Ch} : K(IP(1, 2)) \longrightarrow A^*(IP(1, 2))_{\mathbb{Q}}$  corresponds to the map  $\sigma \mapsto \exp(\mu) = 1 + \mu$  and  $\tau \mapsto \exp(\nu) = 1 + \nu$ .

*Proof.* Since  $(y_0^1)^2 = y_0^2$  and

$$\rho_{\pm}(0, 0)^2 = \frac{1}{2}((y_0^0 + y_0^2) \pm (y_1^0 + y_1^1)),$$

$\{y_0^0, y_0^1, y_0^2, \rho_+(0, 0), \rho_+(0, 0)^2\}$  is a basis for the  $\mathbb{Q}$ -vector space  $K(IP(1, 2))_{\mathbb{Q}}$ . Thus,  $K(IP(1, 2))_{\mathbb{Q}}$  is generated as a ring  $\mathbb{Q}$ -algebra by  $y_0^1$  and  $\rho_+(0, 0)$ . A calculation shows that the relations

$$(y_0^1 - 1)((y_0^1)^2 - 1) = (\rho_+(0, 0) - 1)(\rho_+(0, 0)^2 - 1) = (y_0^1 - \rho_+(0, 0))(\rho_+(0, 0) - 1) = 0$$

hold. A dimension count shows that these are the only relations. Therefore,  $\Phi_+$  is an isomorphism of  $\mathbb{Q}$ -algebras. The previous analysis holds verbatim if  $\rho_+(0, 0)$  is replaced by  $\rho_-(0, 0)$  everywhere.

A similar analysis holds for the Chow theory.  $\square$

**Remark 7.14.** The presentation in the previous proposition yields an exotic integral structure in virtual K-theory and Chow theory. Consider the lattice in  $A^*(IP(1, 2))_{\mathbb{Q}}$

$$\mathbf{A}^{\{0\}}(IP(1, 2)) := \{uc_0^0 \mid u \in \mathbb{Z}\},$$

and

$$\mathbf{A}^{\{1\}}(IP(1, 2)) := \{(u + \frac{v}{2})c_0^1 + \frac{v}{2}c_1^0 \mid u, v \in \mathbb{Z}\}.$$

$\mathbf{A}^*(I\mathbb{P}(1, 2))$  is a subring of the virtual Chow ring  $A^*(I\mathbb{P}(1, 2))_{\mathbb{Q}}$  such that  $\mathbf{A}^*(I\mathbb{P}(1, 2)) \otimes \mathbb{Q} \cong A^*(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ . In fact, the graded ring  $\mathbf{A}^*(I\mathbb{P}(1, 2))$  is isomorphic to

$$\frac{\mathbb{Z}[\mu, \nu]}{\langle \mu, \nu \rangle^2}.$$

Similarly, consider the subring  $\mathbf{K}(I\mathbb{P}(1, 2))$  (not sub- $\mathbb{Q}$ -algebra) of  $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$  generated by  $\{\rho_1(0, 0), \rho_+(0, 0)\}$ . The ring  $\mathbf{K}(I\mathbb{P}(1, 2))$  is isomorphic to

$$\frac{\mathbb{Z}[\sigma, \sigma^{-1}, \tau, \tau^{-1}]}{\langle (\tau - 1)(\tau^2 - 1), (\sigma - 1)(\sigma^2 - 1), (\sigma - \tau)(\tau - 1) \rangle}.$$

(One obtains the same ring  $\mathbf{K}(I\mathbb{P}(1, 2))$  if  $\rho_+(0, 0)$  is replaced by  $\rho_-(0, 0)$  above.) We observe that  $\mathbf{K}(I\mathbb{P}(1, 2)) \otimes \mathbb{Q} \cong K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$  and that  $\mathbf{K}(I\mathbb{P}(1, 2))$  has a natural virtual  $\lambda$ -ring structure which induces the  $\lambda$ -ring structure on  $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ . The  $\lambda$ -positive elements  $\mathbf{P}_1$  of  $\mathbf{K}(I\mathbb{P}(1, 2))$  form the subgroup generated by  $\rho_0(0, 0), \rho_1(0, 0), \rho_+(0, 0)$ . The first virtual Chern class taking  $\tilde{c}^1 : \mathbf{P}_1 \longrightarrow \mathbf{A}^{\{1\}}(I\mathbb{P}(1, 2))$  is a group isomorphism which satisfies

$$\tilde{c}^1(\rho_1(0, 0)^u \rho_+(0, 0)^v) = (u + \frac{v}{2})c_0^1 + \frac{v}{2}c_1^0$$

for all  $u, v$  in  $\mathbb{Z}$ .

**7.2.3. The virtual K-theory and virtual Chow ring of  $\mathbb{P}(1, 3)$ .** We now study the virtual K-theory and virtual Chow ring of  $\mathbb{P}(1, 3)$ . Unlike the case of  $\mathbb{P}(1, 2)$ , the virtual K-theory and virtual Chow rings of  $\mathbb{P}(1, 3)$  differ from the orbifold K-theory and the orbifold Chow rings of the cotangent bundle  $T^*\mathbb{P}(1, 3)$ , respectively. In particular, the inertial pair from the orbifold theory of  $T^*\mathbb{P}(1, 3)$  is Gorenstein but not strongly Gorenstein. We will now describe the  $\lambda$ -positive elements of virtual K-theory of  $\mathbb{P}(1, 3)$ . Unlike the case of  $\mathbb{P}(1, 2)$ , we need to work with  $\mathbb{C}$ -coefficients so that the set of  $\lambda$ -line elements generate the entire virtual K-theory group.

**Remark 7.15.** For the remainder of this section, unless otherwise specified, all products are with respect to the virtual products.

**Proposition 7.16.** *Let  $(K(I\mathbb{P}(1, 3))_{\mathbb{C}}, \star_{virt}, 1 := y_0^0, \tilde{\psi})$  be the virtual K-theory ring with its virtual  $\lambda$ -ring structure. The set of its  $\lambda$ -line elements  $\mathcal{P}_1$  spans the  $\mathbb{C}$ -vector space  $K(I\mathbb{P}(1, 3))_{\mathbb{C}}$ , and, therefore, the virtual K-theory is equal to its core subring. The restriction of the inertial dual  $\mathcal{P}_1 \longrightarrow \mathcal{P}_1$  agrees with the operation of taking the inverse and  $\mathcal{P}_1$  consists of 27 orbits of the action of the translation group  $J_{\mathbb{C}}$ , where each orbit has a unique representative\* in the set  $\{\Sigma_i\}_{i=1}^3 \coprod \{\mathcal{D}_{i,j} \mid i = 1, 2, 3 \text{ and } j = 1, 2\} \coprod \{\mathcal{T}_{i,k} \mid i = 1, \dots, 6 \text{ and } k = 0, 1, 2\}$  given by the following (where  $\zeta_3 = \exp(2\pi i/3)$ ),  $j \in \{1, 2\}$  and  $k \in \{0, 1, 2\}$ :*

$$\begin{aligned} \Sigma_1 &= y_0^0, & \Sigma_2 &= y_0^1, & \Sigma_3 &= y_0^2, \\ \mathcal{D}_{1,j} &= \frac{1}{3}y_0^0 + \frac{1}{3}y_0^1 + \frac{1}{3}y_0^2 - \frac{1}{3}\zeta_3^j y_1^0 + \frac{1}{3}y_1^1 - \frac{1}{3}\zeta_3^{2j} y_2^0 + \frac{1}{3}y_2^1, \\ \mathcal{D}_{2,j} &= \frac{1}{3}y_0^0 + \frac{1}{3}y_0^1 + \frac{1}{3}y_0^2 - \frac{1}{3}y_1^0 + \frac{1}{3}\zeta_3^j y_1^1 - \frac{1}{3}y_2^0 + \frac{1}{3}\zeta_3^{2j} y_2^1, \\ \mathcal{D}_{3,j} &= \frac{1}{3}y_0^0 + \frac{1}{3}y_0^1 + \frac{1}{3}y_0^2 - \frac{1}{3}\zeta_3^{2j} y_1^0 + \frac{1}{3}\zeta_3^j y_1^1 - \frac{1}{3}\zeta_3^j y_2^0 + \frac{1}{3}\zeta_3^{2j} y_2^1, \\ \mathcal{T}_{1,k} &= \frac{1}{3}y_0^0 + \frac{2}{3}y_0^2 + \frac{1}{3}\zeta_3^k y_1^0 + \frac{1}{3}\zeta_3^{2k} y_2^0, \end{aligned}$$

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\*This representative need not be the same as the one defined in Corollary (7.8).

$$\begin{aligned}
\mathcal{T}_{2,k} &= \frac{2}{3}y_0^0 + \frac{1}{3}y_0^2 - \frac{1}{3}\zeta_3^k y_1^0 - \frac{1}{3}\zeta_3^{2k} y_2^0, \\
\mathcal{T}_{3,k} &= \frac{2}{3}y_0^0 + \frac{1}{3}y_0^1 + \frac{1}{3}\zeta_3^k y_1^1 + \frac{1}{3}\zeta_3^{2k} y_2^1, \\
\mathcal{T}_{4,k} &= \frac{1}{3}y_0^0 + \frac{2}{3}y_0^1 - \frac{1}{3}\zeta_3^k y_1^1 - \frac{1}{3}\zeta_3^{2k} y_2^1, \\
\mathcal{T}_{5,k} &= \frac{1}{3}y_0^1 + \frac{2}{3}y_0^2 + \frac{1}{3}\zeta_3^k y_1^0 + \frac{1}{3}\zeta_3^k y_1^1 + \frac{1}{3}\zeta_3^{2k} y_2^0 + \frac{1}{3}\zeta_3^{2k} y_2^1, \\
\mathcal{T}_{6,k} &= \frac{2}{3}y_0^1 + \frac{1}{3}y_0^2 - \frac{1}{3}\zeta_3^k y_1^0 - \frac{1}{3}\zeta_3^k y_1^1 - \frac{1}{3}\zeta_3^{2k} y_2^0 - \frac{1}{3}\zeta_3^{2k} y_2^1.
\end{aligned}$$

**Remark 7.17.** Unlike the case of  $\mathbb{P}(1, 2)$ , the real line elements in  $\mathcal{P}_1$  in  $K(I\mathbb{P}(1, 3))_{\mathbb{C}}$  do not span  $K(I\mathbb{P}(1, 3))_{\mathbb{Q}}$  but it is sufficient to work with an extension of  $\mathbb{Q}$  containing the third roots of unity.

*Proof.* The  $\lambda$ -line elements in  $\mathcal{P}_1$  are calculated by applying the algorithm summarized in Remark (7.7) and by showing that these  $\lambda$ -line elements are invertible. The fact that the elements of  $\mathcal{P}_1$  span  $K(\mathbb{P}(1, 3))_{\mathbb{C}}$  is also a straightforward calculation.  $\square$

**Proposition 7.18.** *Let  $K(I\mathbb{P}(1, 3))_{\mathbb{C}}$  be the virtual  $K$ -theory with its virtual  $\lambda$ -ring structure. We have an isomorphism of  $\mathbb{C}$ -algebras  $\Psi : \mathbb{C}[\sigma^{\pm 1}, \tau^{\pm 1}, \bar{\tau}^{\pm 1}] / \mathbf{I} \longrightarrow K(I\mathbb{P}(1, 3))_{\mathbb{C}}$ , where  $\Psi(\sigma) = \Sigma_1$ ,  $\Psi(\tau) = \mathcal{T}_{1,1}$ , and  $\Psi(\bar{\tau}) = \mathcal{T}_{1,2}$ , where the ideal  $\mathbf{I}$  is generated by the following 10 relations:*

$$\begin{aligned}
\mathcal{R}_1 &:= \sigma^3 - 2\sigma^2 + \sigma - \tau^2 + \tau\bar{\tau} + \tau - \bar{\tau}^2 + \bar{\tau} - 1, \\
\mathcal{R}_2 &:= (\tau - 1)(\tau^2 - \sigma), \quad \bar{\mathcal{R}}_2 := (\bar{\tau} - 1)(\bar{\tau}^2 - \sigma), \\
\mathcal{R}_3 &:= (\tau - 1)(\sigma^2 - \tau), \quad \bar{\mathcal{R}}_3 := (\bar{\tau} - 1)(\sigma^2 - \bar{\tau}), \\
\mathcal{R}_4 &:= \sigma^2 - \sigma\tau - \sigma\bar{\tau} + \tau^2\bar{\tau} - \tau\bar{\tau} + \bar{\tau}^2 - \bar{\tau} + 1, \\
\bar{\mathcal{R}}_4 &:= \sigma^2 - \sigma\tau - \sigma\bar{\tau} + \tau^2 + \tau\bar{\tau}^2 - \tau\bar{\tau} - \tau + 1, \\
\mathcal{R}_5 &:= (\tau - 1)(\sigma\tau - 1), \quad \bar{\mathcal{R}}_5 := (\bar{\tau} - 1)(\sigma\bar{\tau} - 1), \\
\mathcal{R}_6 &:= -\sigma^2 + \sigma\tau\bar{\tau} + \sigma - \tau^2 + \tau\bar{\tau} - \bar{\tau}^2.
\end{aligned}$$

*It follows that  $(\sigma - 1)(\sigma^3 - 1)$  belongs to  $\mathbf{I}$ , which is the relation on the untwisted sector. Furthermore, every element  $K(I\mathbb{P}(1, 3))_{\mathbb{C}}$  can be uniquely presented as a polynomial  $\{\sigma, \tau, \bar{\tau}\}$  of degree less than or equal to 2. In particular, we have*

$$\begin{aligned}
\sigma^{-1} &= -\sigma^2 + \sigma - \tau^2 + \tau\bar{\tau} + \tau - \bar{\tau}^2 + \bar{\tau}, \\
\tau^{-1} &= -\sigma\tau + \sigma + 1, \\
\bar{\tau}^{-1} &= -\sigma\bar{\tau} + \sigma + 1.
\end{aligned}$$

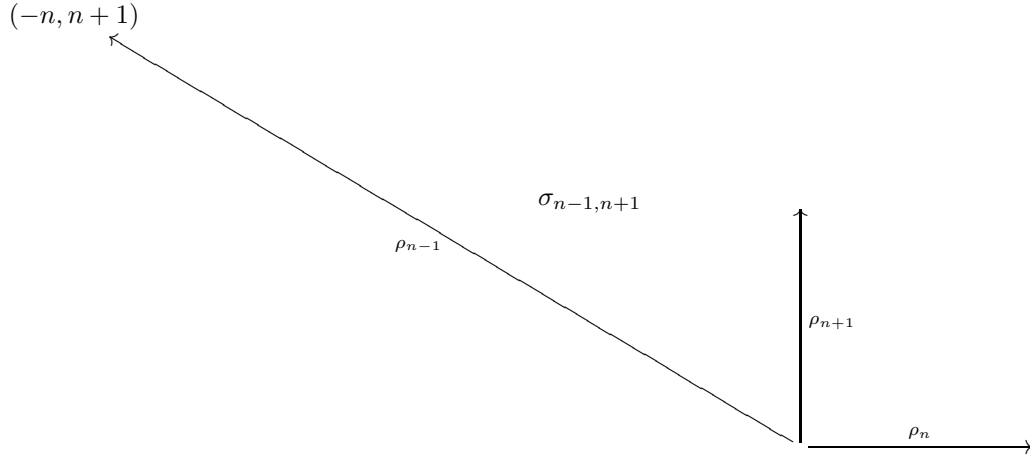
*Proof.*  $K(I\mathbb{P}(1, 3))_{\mathbb{C}}$  is a 10 dimensional  $\mathbb{C}$ -vector space. A direct calculation shows that the set of all monomials in  $\{\sigma, \tau, \bar{\tau}\}$  of degree less than or equal to 2 is a basis of this vector space. The 10 relations correspond to the 10 cubic monomials in  $\{\sigma, \tau, \bar{\tau}\}$ . The expression for the inverses can be verified by direct computation.  $\square$

**Remark 7.19.** The above 10 relations are not algebraically independent.

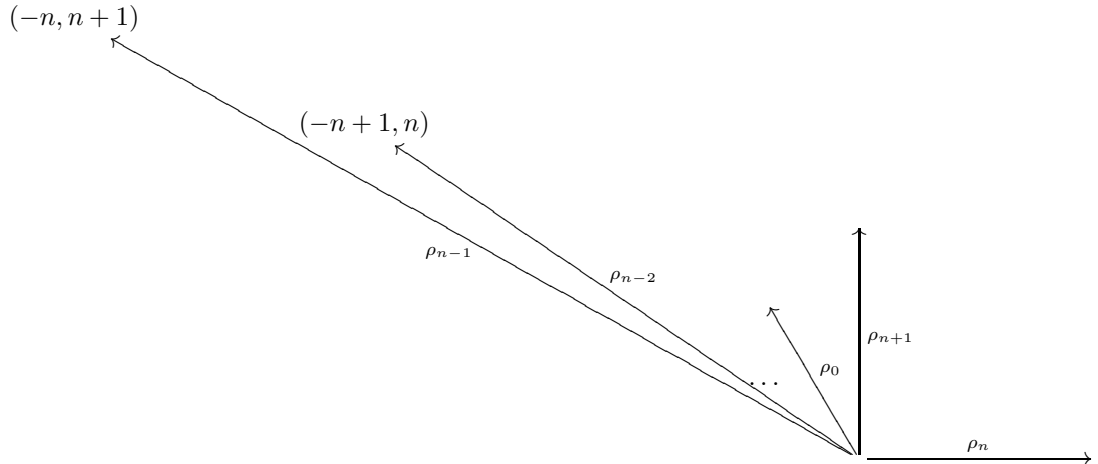
**Remark 7.20.** There is an isomorphism of groups  $\mathcal{P}_1 \longrightarrow (\mathbb{Z}_3)^3 \rtimes J_{\mathbb{C}}$  given by  $\sigma' + j \mapsto ((1, 0, 0), j)$ ,  $\tau' + j \mapsto ((0, 1, 0), j)$ , and  $\bar{\tau}' + j \mapsto ((0, 0, 1), j)$  for all  $j$  in  $J_{\mathbb{C}}$ , where  $\sigma'$ ,  $\tau'$ , and  $\bar{\tau}'$  are the unique elements in the  $J_{\mathbb{C}}$ -orbit containing  $\sigma$ ,  $\tau$ , and  $\bar{\tau}$ , respectively, from Corollary (7.8).

**Remark 7.21.** Restricting  $\Psi$  to  $\mathbb{Z}[\sigma^{\pm 1}, \tau^{\pm 1}, \bar{\tau}^{\pm 1}]/\mathbf{I}$  yields an exotic integral structure on the virtual K-theory  $K(I\mathbb{P}(1, 3))_{\mathbb{C}}$ . The inertial Chern character homomorphism  $\widehat{\mathcal{H}} : K(I\mathbb{P}(1, 3))_{\mathbb{C}} \longrightarrow A^*(I\mathbb{P}(1, 3))_{\mathbb{C}}$  induces an exotic integral structure on virtual Chow theory.

**7.3. The resolution of singularities of  $\mathbb{T}^*\mathbb{P}(1, n)$  and the HKRC.** The cotangent bundle stack to  $\mathbb{P}(1, n)$  is the quotient stack  $[(X \times \mathbb{A}^1)/\mathbb{C}^{\times}]$ , where  $\mathbb{C}^{\times}$  acts with weights  $(1, n, -(n+1))$ . This quotient stack is the toric stack associated to the following simplicial fan, which we denote by  $\Sigma_n$ .



The cone  $\sigma_{n-1, n+1}$  has multiplicity  $n+1$  and the toric resolution of singularities of the toric variety  $X(\Sigma_n)$  is the non-singular toric variety determined by the fan  $\Sigma'_n$ , where we subdivide the cone  $\sigma_{n-1, n+1}$  along the rays  $\rho_0, \rho_1, \dots, \rho_{n-2}$  where  $\rho_i$  is generated by  $-(i+1, i+2)$ .



The toric variety  $Z_n = X(\Sigma'_n)$  is isomorphic to the quotient of  $\mathbb{A}^{n+2} \setminus Z(\Sigma'_n)$  with coordinates  $(x_0, \dots, x_{n+1})$  by the free action of  $(\mathbb{C}^\times)^n$  with weights

$$(\chi_0, \dots, \chi_{n-1}, \chi_0 + 2\chi_1 + \dots + n\chi_{n-1}, -2\chi_0 - 3\chi_1 - \dots - (n+1)\chi_{n-1}),$$

where  $\chi_i$  is the character of  $(\mathbb{C}^\times)^n$  corresponding to the  $i$ -th standard basis vector of  $\mathbb{Z}^n$  and  $Z(\Sigma'_n) = V(x_2x_3 \dots x_{n+1}, x_0x_3 \dots x_{n+1}, x_0x_1x_4 \dots x_{n+1}, \dots, x_0 \dots x_{n-2}x_nx_{n+1}, x_0 \dots x_{n-1}, x_1x_2 \dots x_n)$ . The  $K$ -theory and Chow rings of this toric variety are readily calculated, yielding

$$K(X(\Sigma'_n)) = \frac{\mathbb{Z}[\chi_0, \chi_0^{-1}, \dots, \chi_{n-1}, \chi_{n-1}^{-1}]}{\langle \text{eu}(\chi_0), \dots, \text{eu}(\chi_{n-1}) \rangle^2}$$

and

$$A^*(X(\Sigma'_n)) = \frac{\mathbb{Z}[t_0, t_1, \dots, t_{n-1}]}{\langle t_0, t_1, \dots, t_{n-1} \rangle^2},$$

where  $t_i = c_1(\chi_i)$

**Proposition 7.22.** *Let  $Z_n$  be the crepant resolution of singularities of the moduli space of  $T^*\mathbb{P}(1, n)$  indicated by the toric diagram above. Then for  $n = 2, 3$  there are isomorphisms of augmented  $\lambda$ -rings  $\widehat{K}(\mathbb{P}(1, n))_{\mathbb{C}} \rightarrow K(Z_n)_{\mathbb{C}}$  where the  $\widehat{K}(\mathbb{P}(1, n))_{\mathbb{C}}$  has the inertial  $\lambda$ -ring structure described above.*

*Proof.* We have calculated  $K(\mathbb{P}(1, 2))_{\mathbb{C}}$  and  $K(\mathbb{P}(1, 3))_{\mathbb{C}}$  and in both cases we obtain an Artin ring which is a quotient of a coordinate ring of a torus of rank 2 and 3, respectively. The inertial augmentation ideal corresponds to the identity in the corresponding torus. Thus for  $n = 2, 3$  the ring  $\widehat{K}(\mathbb{P}(1, n))_{\mathbb{C}}$  is simply the localization of  $K(\mathbb{P}(1, n))_{\mathbb{C}}$  at the corresponding maximal ideal. Direct calculation shows that  $\widehat{K}(\mathbb{P}(1, 2))_{\mathbb{C}} = \mathbb{C}[\sigma, \sigma^{-1}, \tau, \tau^{-1}] / \langle \text{eu}(\sigma), \text{eu}(\tau) \rangle^2$  and  $\widehat{K}(\mathbb{P}(1, 3))_{\mathbb{C}} = \mathbb{C}[\sigma, \sigma^{-1}, \tau, \tau^{-1}, \bar{\tau}, \bar{\tau}^{-1}] / \langle \text{eu}(\sigma), \text{eu}(\tau), \text{eu}(\bar{\tau}) \rangle^2$ , which are readily seen to be isomorphic as  $\lambda$ -rings to  $K(Z_2)_{\mathbb{C}}$  and  $K(Z_3)_{\mathbb{C}}$ , respectively.  $\square$

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